UNIVERSIDAD AUTÓNOMA
DEL ESTADO DE MÉXICO

Faculty of Engineering

Mechanical circuits-based modelling for mechanical dynamic systems

Thesis

To obtain the grade of Master of Engineering Science

Daniela Montserrat Ruiz López

Academic Tutor: Dr. Juan Carlos Ávila Vilchis
Assistant Tutor: Dr. Martín Carlos Vera Estrada
Assistant Tutor: Dra. Martha Belem Saldivar Márquez

Contents

List of Figures

1 Abstract

The mathematical model of mechanical systems can be obtained through conventional methodologies as free body diagram, Lagrange equations, and electromechanical analogies, among others. These methodologies contemplate the mechanical systems excited by force sources. However, the mechanical systems can be excited by displacement sources too where conventional methodologies fail or represent long-winded procedures.

A methodology to cope with mechanical systems excited by displacement sources is introduced. This approach is based on the mechanical circuit representation, a scheme that allows that exposes the interconnections and interactions of the elements of the mechanical system. Some of the conventional methodologies cannot apply when displacement sources are present. If a comparison is established between the conventional methodologies that can work with displacement sources, the methodology developed in this work allows one to obtain the mathematical model of a system faster than using them. The mechanical circuit approach let calculate the forces that would provoke the displacement of each mechanical element in the system directly, as will be shown later.

This thesis is distributed as follows. The research proposal is presented in Section 2, where the bibliographic review of this work is reported. Furthermore, in Section 2, the hypothesis, the objectives, and the methodology employed in this project (research work) are presented.

Section 3 presents the scientific paper which was went to the journal "Mathematical and Computer Modelling of Dynamical Systems", whose title is "Linear Mechanical Systems with Displacement Sources: Mathematical Modeling". This paper establishes a methodology to obtain the mathematical model of mechanical systems excited by displacement forces. The paper settles the displacement and accelerations laws, proving two theorems and providing three case studies. The last part of this work includes some appendices where concepts developed during this research are shown, these concepts are the base of a methodology to obtain the mathematical model of mechanical systems contemplating more elements and interconnections than the case reported in the paper. The appendices show information research that can be enlarged to be reported in a journal as future work. The appendices pro-

vide the relations to convert from delta to wye mechanical interconnections and vice versa, the representation of rotational mechanical elements in a circuit way, and the steps to get a mathematical model of systems with these elements. In these appendices is presented how to obtain the equations of motion of systems with super mechanical meshes and super mechanical nodes and also, an example of the previous situation.

2 Research proposal

2.1 Bibliographic review

As the objective of this project is to establish a methodology to obtain the mathematical model of a dynamic system (specifically, of a mechanical system), the next section provides information about what exactly a mathematical model is.

Once the mathematical modelling review is complete, it is relevant to mention why a mathematical model is important and its use in the control area.

There are several methodologies to obtain the mathematical model of a mechanical system. These conventional methodologies need to be analysed in order to compare their advantages and disadvantages. This analysis allows to find an opportunity area for the development of a new methodology that simplifies the process when a mechanical system evolves in general movement (translational and rotational movement).

A different methodology for obtaining a mathematical model needs to cover the aspects that conventional methodologies do not include. Some of these conventional methodologies are, briefly, described below.

2.1.1 Theoretical framework

Modelling of mechanical systems

Mathematical modelling of a dynamic system, involves establishing the equations that describe its behaviour. The importance of modelling the behaviour of a system has several applications. For instance, to simulate a plant behaviour under certain conditions and thus, to know in advance how it will react to external signals. The mathematical modelling is essential to analyse the operation of the system, and by the manipulation of certain variables from its dynamic equations the system can be forced to behave in the desired way, this is known as control.

2.1.2 Conventional methods for mechanical systems modelling

This section presents some conventional methods used to obtain a mathematical model of a mechanical system. Firstly, the free-body diagram, which is based on the Newton's second law, is addressed. Then, the grounded chair method, where the configuration of the mechanical elements play an important role to in obtaining the equations of motion, is discussed. Next, the direct and indirect electromechanical analogies are described, specifying the differences between them. Finally, the Lagrange equations and the bond graph approach, that are based on the energy of the systems, are presented and some aspects of the absolute motion analysis, that allows one to obtain the mathematical model from the geometry of the mechanical system, are discussed.

Free-body diagram

Isaac Newton's second law establishes that the acceleration of a body is a consequence of the action of a net force acting on it [5]. The forces acting on a body are shown in a free-body diagram, they are added and equated to the rate of change of its momentum [6].

Through this method it is possible to analyse a mechanical system in a general way, omitting the internal forces effects between its components or taking them into account and thus, obtaining the mathematical model of each of its parts. However, in free-body diagrams the influence of some elements on others is not always clarified.

Grounded chair

The aim of this method is to obtain the equations of motion according to the configuration of the mechanical elements, that are part of the mechanical system, either in a series or in a parallel interconnection. To visualize this interconnection it is necessary to make a scheme where all the masses are placed near to the ground [1].

To draw the mechanical system shown in Figure 1a in a grounded chair way, the displacement x needs to be at the top of the diagram with the force f acting on it, followed by the displacement y (see Figure 1b). Then, the mechanical elements are introduced according to the variables that describe their movement (between x and the ground or between x and y) respecting their series or parallel distribution (see Figure 1c). Observe that the displacement receiving the external force action is located at the top of the diagram (in this case, the x displacement).

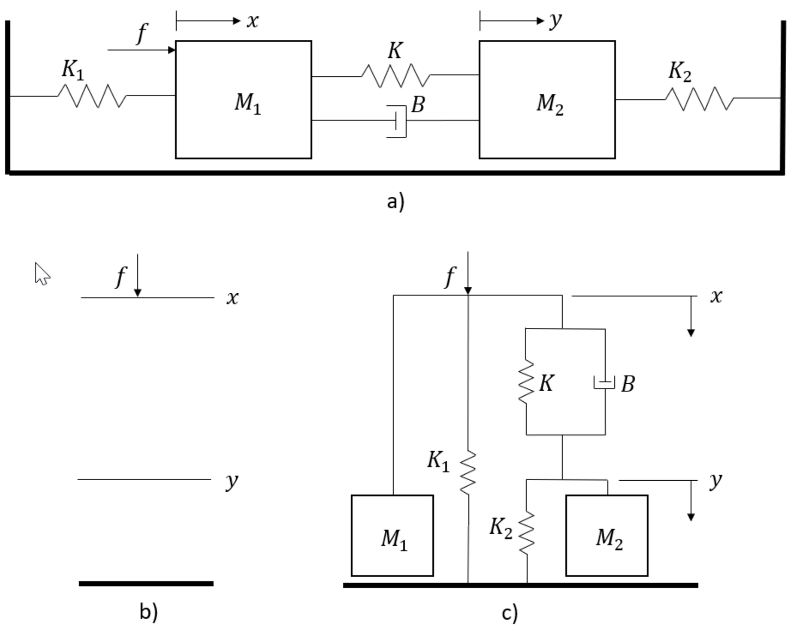


Figure 1: a)Mechanical system, b) Descending displacements, c) Grounded chair representation (modified from [1]).

One of the disadvantages of this method occurs when two or more external forces affect the system since the number of calculations will increase according to the number of forces.

Direct electrical analogy

The differential equation associated to a series resistor-inductor-capacitor circuit excited by a voltage source or to a mass-spring-damper parallel longitudinal mechanical system perturbed by an external force, has the form (1).

$$a\ddot{q} + b\dot{q} + cq = E \tag{1}$$

where, respectively, q represents the electric charge or the linear displacement, a the inductance or the mass, b the electrical resistance or the damping constant, c the reciprocal of the capacitance or the spring constant and E the voltage source or the external force [1]. In this document, time derivatives are denoted by dots ($\dot{q} = dq/dt$, $\ddot{q} = d^2q/dt^2$, etc.).

The direct electromechanical analogy relates, from the energetic point of view, similar forms of energies: it associates the kinetic energy of a mass (speed) with the kinetic energy of an electric charge (electrical current). This analogy associates electrical and mechanical impedances [7].

Indirect electrical analogy

The differential equation associated to a series resistor-inductor-capacitor circuit excited by a voltage source or to a mass-spring-damper series longitudinal mechanical system perturbed by an external force f, has the form (2).

$$a\ddot{w} + b\dot{w} + cw = P \tag{2}$$

where, respectively, P represents the voltage or the acceleration, a the inductance or the reciprocal of the spring constant, b the resistance or the reciprocal of the damping constant, c the reciprocal of the capacitor or the reciprocal of the mass and w the electric charge or the force [1].

The indirect electrical to mechanical analogy analogy receives that name due to the fact that it associates the kinetic energy of the mass from a mechanical system with the potencial energy stocked in the capacitor from an electrical circuit so, the electrical admittance is related with the mechanical impedance [7].

Based on these similarities, a mechanical system can be represented in the form of an electric circuit by replacing mechanical components with their equivalent electrical elements. Once the equations of the system are obtained, based on circuit theory, it is required to return the electrical components to their analogue mechanical components. The need to switch from mechanical to electrical components and vice versa in this analogies can be seen as a disadvantage in several cases where the complexity of the system is significant.

Lagrange equations

Lagrangian mechanics allows one to obtain, in terms of generalized coordinates, the differential equations of a mechanical system based on the variations of kinetic and potential energies. The use of the Lagrangian formulation is based on the principle of work and energy. Therefore, it is more useful to employ Lagrange equations than free-body diagrams when working with multiple degrees of freedom systems [8]. This method requires the use of generalised coordinates which are defined in function of the degrees of freedom of the system [9] and the familiarization with energy concepts.

Bond graph method

Mathematical modelling of dynamic systems using bond graph technique is based on the exchange of energy (flows and efforts) among the elements of a system [2]. This exchange is represented by joints and the components by vertices called ports (inductive, capacitive, resistive, sources or sinks, transformers or gyrators). The joints have a half arrow designating the flow direction and a perpendicular line called causality, which represents the direction of the effort (see Figure 2).

The union graphs are an abstraction of the original system, a graphical methodology through which it is possible to obtain the state equations of mechanical, electrical, hydraulic and thermal systems. This method requires familiarization with energy concepts to be able to model dynamic systems. It also requires a change of variables in order to be applied [2].

Absolute motion analysis

Absolute motion analysis allows obtaining the mathematical model of a mechanical system considering its geometry. If a point is moving along a straight line s that rotates θ degrees, s and θ can be related based on the geometry of the problem and the equation of motion can be obtained [10]. The com-

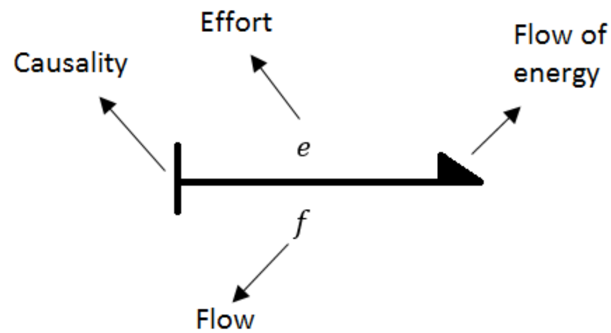


Figure 2: Common binding nomenclature in a bond graph (modified from [2]).

plexity of this method depends on the geometry of the system.

2.2 Background and state of the art

In this section the relationship between mechanical systems and electrical circuits through the relation between voltages and forces law and the currents and velocities law is exposed. These associations sustain the possibility of obtaining the mathematical model of a mechanical system using its equivalent mechanical circuit.

Electromechanical analogies have played an important role as, for instance, a better design of electromechanical transducers telephones, the development of networks theory and its use in electric wave systems [11]. One of their advantages is to analyse, at the same time, the mechanical and the electrical elements to know how they interact with each other, specially in transducers.

The study of electromechanical analogies allowed Malcom Smith [12] to find an error in associating the mass as equivalent to the electrical capacitor and then, proposes the "inerter", a mechanical component equivalent to the electrical capacitor that has not been included within the elements that form the mechanical systems in the conventional methodologies and that this project will consider. Furthermore, the electromechanical analogies for mechanical elements with four terminals are presented as an antecedent in a circuit.

Then, it is introduced the methodology that this project is seeking to expand: an algorithm with the purpose of obtaining the mechanical system mathematical model from a mechanical circuit representation [13]. This methodology contemplates masses, springs and dampers as mechanical elements and forces as sources of excitation. In such a way, the mechanical circuit constitutes the base of this investigation.

Relationship between mechanical systems and electrical circuits

The current law for electrical circuits is analogous to the forces law for mechanical systems. The Kirchhoff's current law establishes that the sum of the electrical currents entering a node is equal to the sum of the currents leaving it [14]. The forces law [15] for mechanical systems establishes that the sum of the forces entering in a union is equal to the sum of forces that come out.

Similarly, the voltage law is analogous to the velocities law. The Kirchhoff's voltage law states that the algebraic sum of all the voltages around any closed loop in an electrical circuit is equal to zero [14]. The velocities law [15] sustains that the sum of the increases of speed in a closed circuit is equal to the sum of the speed that decreases in it.

Firestone [15] presents direct relationships between the currents and forces laws as well as between the voltages and velocities laws.

Inerter

Before 2002 the indirect electromechanical analogy compared the mass with a capacitor, but they are not completely analogous due to the fact that the capacitor has two moveables terminals whereas the mass has only one and the other terminal must always be connected to the ground. In 2002 Malcolm Smith [12] found a mechanical device which he named "inerter", equivalent to a capacitor when it is not connected to the ground. Taking into account the "inerter", the mentioned analogy becomes accurate. If a mechanical system containing springs, masses, inerters and dampers is excited by an external force, the "inerter" which is a mechanical element of two moveables terminals, will produce an acceleration between the terminals proportional to the force applied to them.

In 2008, Smith explains the advantages of using an inerter in a vehicle suspension [12]. Then, it was implemented in a formula 1 car, run by Kimi Raikkonen in 2005, noting an increase in the potential performance of the vehicle [16]. Recent studies show that the inerter reduces the natural frequencies of a MDOF (Multiple Degrees Of Freedom) system when it increases its inertance [17], eliminates oscillations of high buildings [18] and reduces dynamic vibrations [19], among other applications.

Mechanical circuit with elements of four terminals

Based on the electrical analogies, the engineer Adrián Kisielewsky [20] develops a new convention to draw "mechanical diagrams". He establishes the relationship between the next passive mechanical elements with four terminals: pulleys, levers, zips, ideal and endless gears and an ideal voltage transformer.

The mechanical diagrams presented by Adrián Kisielewsky allow representing the mechanical elements with four terminals in a circuit way.

Mechanical circuit

This method models linear mechanical systems based on a variation of Newton's second law, the D'Alembert's principle and the currents Kirchhoff's law, redrawing the system in a similar way as an electrical circuit but without the use of electrical to mechanical analogies [13].

The mechanical circuit is drawn by considering the nodes as the points where one or more mechanical elements (mass, spring or damper) are joined. That elements are considered reactive elements. The forces are considered as excitation sources. Then, the force law is applied to obtain the equations of motion [13].

One of the advantages of obtaining the mathematical model from a mechanical circuit is that any number of forces can be analysed without applying the superposition principle [13].

2.3 Problem statement

Conventional methods for mathematical modelling of mechanical systems do not contemplate the mechanical element "inerter" and generally the systems are disturbed by external forces but not by external displacements.

Obtaining the equations of motion through the mechanical circuit approach allows one to directly see the relationship between mechanical elements (dampers, springs, masses and inerters) and their effects. The advantage and difference between the mechanical circuit approach and the electrical to mechanical analogies is that the first one does not require to change from mechanical to electrical components, thus it allows obtaining the equations from the original devices.

In view of the previously presented information, this project concerns the problem of modelling mechanical systems with the mechanical circuit approach mentioned by Vera [13], taking into consideration a general case of movement with different configurations (serial, parallel, delta or wye) and different elements (masses, springs, dampers, inerters, gears, pulleys or levers).

2.4 Hypothesis

By the use of a new modelling methodology, it is possible to obtain the equations of motion of mechanical systems in a simpler way (less calculations and fewer steps) compared to conventional methodologies. This new methodology will consider more general cases and elements than the others mentioned in this research proposal.

2.5 Objectives

2.5.1 General objective

To establish a methodology for the mathematical modelling of mechanical systems based on mechanical circuits seeking a simpler way to obtain the equations of motion.

2.5.2 Particular objectives

- To detect different types of linear mechanical systems

- To define the methodology steps for the different types of mechanical systems

- To define the notation for mechanical circuits

- To establish the rules for drawing a mechanical circuit from a mechanical system

- To establish when to use the forces or velocities laws

- To establish the procedure for obtaining the mathematical model of a mechanical circuit with general movement

- To validate the obtained methodology by a mathematical proof

2.6 Scope and limitations

This project concerns the modelling of linear mechanical systems with general motion (translation, rotation or both).

To generate pure rotational movement, the elements that can be included in a mechanical system are: rotational springs, rotational dampers, moments of inertia, rotational inerters, pulleys, levers and gears.

To generate pure translational movement, the elements that can be included in a mechanical system are: translational springs, translational dampers, masses and translational inerters.

The interconnections between mechanical elements that this methodology will consider are series, parallel, delta, wye and combinations of them.

The excitation signals to be considered in this methodology can be classified in three main groups as it is shown below. The two first categories correspond to conventional excitation signals while the last one is a less known kind of excitation.

- **Force sources**. These sources are related to translational movement.
- **Torque sources**. Rotational movement is generated by these sources.
- **Displacement sources**. The effects of these sources will be analysed in this project for both translational and rotational movement.

2.7 Proposed methodology

The papers published until now indicate that there is not a methodology that contemplates the aspects that this project will consider, but it is necessary, during the whole investigation, a constant review of papers about the modelling of mechanical systems with the purpose of detecting methodologies that involve the mechanical circuit approach to obtain the equation of motion of a system. If an algorithm to obtain the mathematical modelling of a mechanical system using the mechanical circuit is found, this project will evaluate in order to to improve it or redefine the investigation.

The first step to approach the aim of this project is the identification of the types of mechanical systems to be considered since the methodology might contemplate different cases. The aspects that need to be known are: the types of movement, the types of excitation sources, and the mechanical elements and their interconnections.

The mentioned aspects need to be grouped, joining the components that share similar characteristics, to contemplate the system from a general to a particular case. Then a particular notation will be defined giving a name to the formed groups.

To establish the steps to obtain the mathematical model of a mechanical system, it is necessary to group some repeating steps in loops, define variables, parameters and conditions to be satisfied.

After this, it is necessary to define how to identify the types of interconnections between the mechanical elements such as parallel, series, delta and wye interconnections in order to define which one of them will be considered as node, a mesh, a super node or a super mesh and to graphically stablish the frame of reference. All this will be necessary to define the procedure to draw the system as a mechanical circuit.

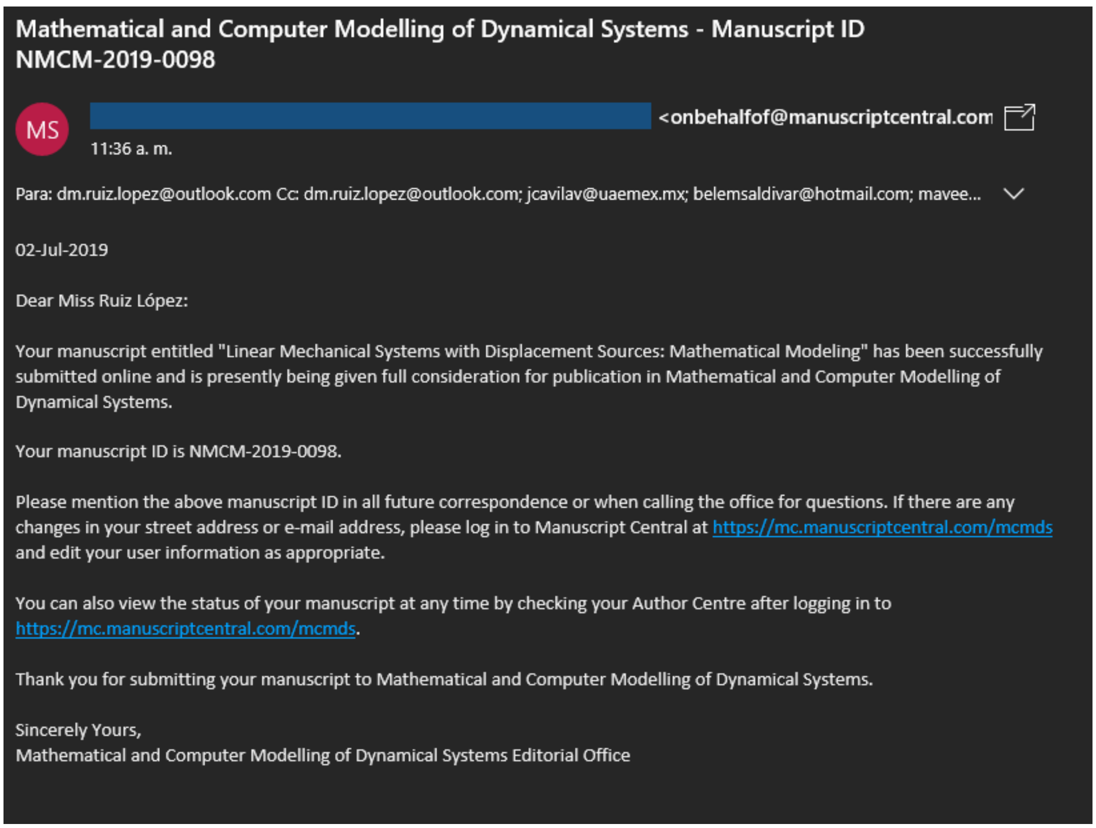
Within the procedure to redraw the system as a mechanical circuit it is fundamental to contemplate that the interaction of the rotational elements with others depends on their constant measures, for instance the pulleys diameter, the levers arms and the number of teeth of the gears. This project will develop a way to graphically represent, in the mechanical circuit these constant parameters.

To define the algorithm to obtain the mathematical model of a mechanical circuit it will be necessary to specify, that non-linear systems are out of the scope of this methodology. Then, the forces and velocities laws have to be defined as well as the situations when these laws are used. The steps and conditions to get the mathematical model of a mechanical circuit in general movement will be established.

A mathematical proof will be proposed in order to validate the methodology, with the purpose to sustain its effectiveness. Once it has been proven that the proposed methodology is suitable for mechanical systems with the specifications mentioned above, the investigation report will be written.

3 Scientific paper

In this section is presented the scientific paper "Linear Mechanical Systems with Displacement Sources: Mathematical Modeling" that was submitted to the journal "Mathematical and Computer Modelling of Dynamical Systems" (see Figure 3).



Mathematical and Computer Modelling of Dynamical Systems - Manuscript ID NMCM-2019-0098

MS [redacted] <onbehalfof@manuscriptcentral.com

11:36 a. m.

Para: dm.ruiz.lopez@outlook.com Cc: dm.ruiz.lopez@outlook.com; jcavilav@uaemex.mx; belemsaldivar@hotmail.com; mavee...

02-Jul-2019

Dear Miss Ruiz López:

Your manuscript entitled "Linear Mechanical Systems with Displacement Sources: Mathematical Modeling" has been successfully submitted online and is presently being given full consideration for publication in Mathematical and Computer Modelling of Dynamical Systems.

Your manuscript ID is NMCM-2019-0098.

Please mention the above manuscript ID in all future correspondence or when calling the office for questions. If there are any changes in your street address or e-mail address, please log in to Manuscript Central at https://mc.manuscriptcentral.com/mcmds and edit your user information as appropriate.

You can also view the status of your manuscript at any time by checking your Author Centre after logging in to https://mc.manuscriptcentral.com/mcmds.

Thank you for submitting your manuscript to Mathematical and Computer Modelling of Dynamical Systems.

Sincerely Yours,
Mathematical and Computer Modelling of Dynamical Systems Editorial Office

Figure 3: Scientific paper submitted to the journal "Mathematical and Computer Modelling of Dynamical Systems"

The present thesis is orientated to obtain a master degree, through the specialized article graduation option. Considering the articles 57, 59 and 60 of the REA (Reglamento de Estudios Avanzados) of the UAEMéx (Universidad Autónoma del Estado de México).

Linear Mechanical Systems with Displacement Sources: Mathematical Modeling

Daniela Ruiz[a,c], Juan Carlos Ávila Vilchis[a,c,*], Belem Saldivar[a,b], Martín Carlos Vera Estrada [a]

[a]Universidad Autónoma del Estado de México, Facultad de Ingeniería, Cerro de Coatepec S/N, Ciudad Universitaria C.P. 50100, México,
[b]Cátedras CONACyT ,

[c]These authors contributed equally to this work.

*corresponding author. E-mail: jcavilav@uaemex.mx

4 General conclusions and future work

The mechanical circuit approach is used to represent a mechanical system in a circuit way, constructing nodes and meshes. Then, the equations of motion of the system are obtained by applying the force or the displacement law.

The number of steps and calculus to obtain the mathematical model of a system, using the approach proposed in this work, are fewer than using conventional methodologies. This is because when the force or displacement law is applied to a mechanical circuits, the elements that are enduring the same displacement or the same force are analysed together.

In this work the approach of mechanical circuits excited by forces and analised by nodes proposed by [13] is expanded. The paper reported in this work deals with displacement sources where a methodology to cope with this kind of excitations is introduced through a mesh analysis.

In the appendices of this document, it is established how to cope with other cases. Appendices show the bases to draw mechanical circuits with rotational mechanical elements, the inerter element and delta or wye configurations.

These information was not reported in the mentioned paper but, as future work, it can be expanded adding case studies and detailed explanations. With the concepts developed in this work, a general methodology to model many different linear mechanical systems can be visualized. But, the author does not discard the possibility of finding more components, interconnections and laws that could be generated or established for linear mechanical systems.

References

[1] F. Raven, *Series in mechanical engineering.* McGraw-Hill, 1978.

[2] D. E. Fuentes Herrera *et al.*, *Uso del método de los gráficos de unión para la solución de problemas de elastodinámica: Una propuesta Metodológica.* PhD thesis, Universidad Nacional de Colombia-Sede Bogotá, 2015.

[3] C. K. Alexander and M. no Sadiku, "Electric circuits," *Transformation*, vol. 135, pp. 4–5, 2000.

[4] N. Makris and G. Kampas, "Seismic protection of structures with supplemental rotational inertia," *Journal of Engineering Mechanics*, vol. 142, no. 11, p. 04016089, 2016.

[5] H. D. F. Young, R. A. F. Flores, *et al.*, *Sears-Zemansky Física universitaria*, vol. 1. Pearson Educación, 2009.

[6] H. Goldstein, *Classical mechanics.* Pearson Education India, 2011.

[7] P. Herzog and G. Penelet, "Section 2.2 : Electromechanical analogies," tech. rep., The University of Maine.

[8] S. G. Kelly, *Mechanical vibrations: theory and applications.* Cengage learning, 2012.

[9] M. M. Ayala and J. E. Barragan, "Representación geométrica de las coordenadas generalizadas en la mecánica hamiltoniana," *Lat. Am. J. Phys. Educ. Vol*, vol. 7, no. 4, p. 616, 2013.

[10] R. C. Hibbeler, *Engineering Mechanics: Dynamics.* Prentice-Hall, 2011.

[11] P. Gardonio and M. Brennan, "On the origins and development of mobility and impedance methods in structural dynamics," *Journal of Sound and vibration*, vol. 249, no. 3, pp. 557–573, 2002.

[12] M. C. Smith, "Synthesis of mechanical networks: The inerter," *IEEE Transactions on Automatic Control*, vol. 47, no. 10, pp. 1648–1662, 2002.

[13] M. Vera Estrada, "Nuevo enfoque en el modelado de sistemas mecánicos: circuitos mecánicos," *Trigésima reunión internacional de verano de potencia, aplicaciones industriales y exposición industrial. IEEE Sección México.*, 2011.

[14] C. K. Alexander and M. N. Sadiku, *Fundamentos de circuitos eléctricos (5a.* McGraw Hill Mexico, 2013.

[15] F. A. Firestone, "A new analogy between mechanical and electrical systems," *The Journal of the Acoustical Society of America*, vol. 4, no. 3, pp. 249–267, 1933.

[16] M. Z. Chen, C. Papageorgiou, F. Scheibe, F.-C. Wang, and M. C. Smith, "The missing mechanical circuit element," *IEEE Circuits and Systems Magazine*, vol. 9, no. 1, 2009.

[17] M. Z. Chen, Y. Hu, L. Huang, and G. Chen, "Influence of inerter on natural frequencies of vibration systems," *Journal of Sound and Vibration*, vol. 333, no. 7, pp. 1874–1887, 2014.

[18] A. Giaralis and F. Petrini, "Wind-induced vibration mitigation in tall buildings using the tuned mass-damper-inerter," *Journal of Structural Engineering*, vol. 143, no. 9, p. 04017127, 2017.

[19] D. Pietrosanti, M. De Angelis, and M. Basili, "Optimal design and performance evaluation of systems with tuned mass damper inerter (tmdi)," *Earthquake Engineering & Structural Dynamics*, vol. 46, no. 8, pp. 1367–1388, 2017.

[20] A. Kisielewsky, "Sistemas electromecánicos," 2013. Accessed: 2017-05-06.

[21] K. Ogata and Y. Yang, *Modern control engineering*, vol. 4. Prentice hall India, 2002.

[22] R. Hibbeler, *Engineering Mechanics Dynamics*. Pearson Prentice Hall, Upper Saddle River, NJ, 2009.

[23] E. E., ed., *Rotations*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1998.

[24] E. Okuno and L. Fratin, *Rotations*, pp. 59–75. New York, NY: Springer New York, 2014.

[25] R. G. Budynas and J. K. Nisbett, *Diseño en ingeniería mecánica de Shigley (9a.* McGraw Hill Mexico, 2012.

[26] H. Young, R. Freedman, F. Sears, and M. Zemansky, "Física universitaria, vol. 1," *Naucalpan de Juárez-Mexico: Pearson Educación de México, SA*, 2009.

[27] V. Acosta, C. L. Cowan, and B. J. Graham, *Curso de física moderna.* No. 530 A258C., Harla, 1975.

Appendices

A Basic concepts

In this appendix, fundamental concepts are presented. Several of these fundamental concepts are based on electrical circuits theory and will help to define essential notions of mechanical circuits. In this section are introduced fundamentals of classical mechanics, linear and non-linear systems and the types of interconnection between elements in a system.

A.1 Linear and non-linear systems

A.1.1 Identification of linear and non-linear systems

The superposition principle sustains that if a system is being affected by two forces, the final response of the system will be equal to the sum of the system response when each force acts separately. If the superposition principle describes the response of a system to more than one force, then is a linear system; otherwise, it is a non-linear system [21].

The modeling method developed in this work do not contemplate the mathematical modeling for non-linear systems, so it is essential to identify this kind of systems to discard the use of the methodology.

A.2 Rotational and translational movement

A.2.1 Rotation

The rotation is defined as the circular movement of the particles of a rigid body, around a fixed axis [22]. Turning, each point (except the rotating center) occupies other position, different to the original. The new and last position will have a one-to-one relation [23]. The mass of the body and how it is distributed around the rotation axis are characteristics of the body that contribute or hinder its state of rotation [24].

A.2.2 Translation

Consider a body in motion. If a line in the body keeps parallel during the movement from its initial orientation, then the body is in translation. There are two types of translation, rectilinear and curvilinear, the first one occurs when the trajectories of two points in the body describe parallel lines, the last one when the trajectories are performed along curved lines [22].

A.2.3 General plane motion

The combination of translation and rotation movement is known as general plane motion, considering that the rotation is carried out on a perpendicular axis from the plane where the translation occurs [22].

A.3 Dimensions of rotational mechanical elements

A.3.1 Pulleys

Ideally, the ratio of the angular velocity between the driven pulley and driving pulley is equal to the ratio of their diameters. However, in practice, there is a slip and creep between bands and pulleys, so, the angular speed ratio between two connected pulleys does not often correspond to the ratio of their diameters, except when it comes to synchronization bands [25].

In order to obtain the mathematical model of pulleys, the bands will be represented as a spring with a specific coefficient of rigidity.

A.3.2 Gears

The coupling of gears tooth has a direct relationship with the angular velocity of the driving shaft and the driven shaft. In a gear train, the speed of the last gear (n_l) is calculated by the following expression [25]:

$$n_l = e * n_f$$

where:
n_f is the speed of the first gear
e is the value of the train

The value of the train is positive if the first and last gear rotate in the same direction, it is calculated as follows [25]:

$$e = \frac{product\ of\ the\ number\ of\ teeth\ of\ the\ drive\ gears}{product\ of\ the\ number\ of\ teeth\ of\ the\ driven\ gears}$$

A.3.3 Levers

If the axis of rotation of a body passes by the point O, A is a plane on the body perpendicular to the axis of rotation and F is a force acting on the body in the direction of the plane A, the torque is defined as $\tau = F * l$ where l is the perpendicular distance between the line of action of the force and the point O. l receives the name of the lever arm [26].

A.4 Excitement sources

The equilibrium state of a body is present when it has constant or zero velocity. A force can be defined as a vector that changes the equilibrium state of a body. The forces are created when there is an interaction between a body and the environment that surrounds it or between a body and another body [26]. As external displacements change the equilibrium state of a body, the external forces and displacements will be considering as excitement sources to a system in this work.

A.5 Reference frame

If a body is able to move along a reference frame at constant velocity without forces acting on it, then, this frame is known as inertial reference frame. Classical mechanics is totally valid within an inertial frame [26].

Analysing relative movements is convenient to use a system of perpendicular coordinates to describe the movement of all bodies. The orthogonal coordinates guarantee that they do not depend on each other [27]. For the mechanical circuit under consideration, the inertial reference frame will be represented by a node of reference.

A.6 Super nodes and super meshes

A super node in electrical circuits is considered a node that includes a voltage source and two nodes. If a nodal analysis is being used to obtain the equations that describe the movement of an electrical circuit, the Kirchhoff's currents law needs to be applied to the circuit considering a super node enclosing the voltage source. Then, the Kirchhoff's voltages law is applied to the mesh where the voltage source is [3].

To solve a mechanical circuit by nodal analysis when it is present a displacement source, the use of a super node will be necessary. In order to obtain the equation of motion of the system, the forces law is applied taking into account the super node enclosing the displacement source. Then the velocities law to the mesh is employed where the displacement source is placed.

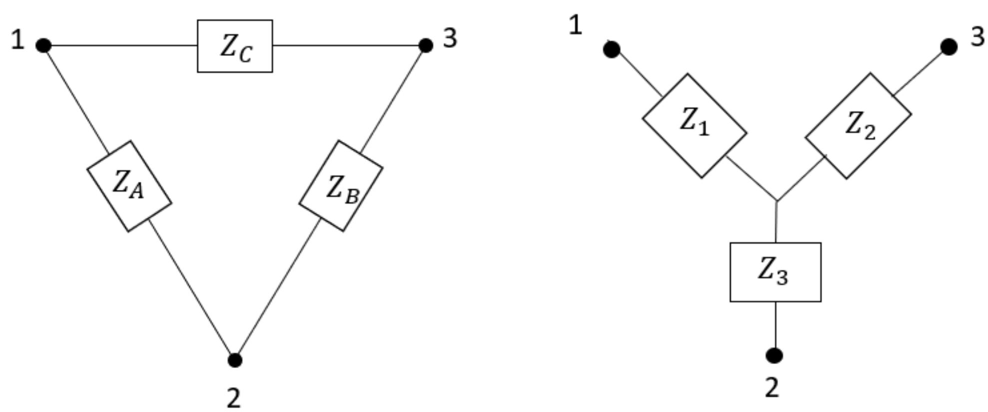
If the mesh analysis is applied to an electrical circuit that contains current sources, a supermesh must be considered. A supermesh is the contour of two joined meshes, omitting the common branch in both, where a current source is placed. The Kirchhoff's voltages law is applied to the supermesh and the Kirchhoff's currents law to the node next to the current source [3].

In a mechanical circuit, analogously the velocities law is applied to a super mesh and the forces law to the node next to the force source.

B Wye and Delta configurations in mechanical systems

Commonly, mechanical elements present series or parallel interconnection. However, a mechanical system may not have its elements interconnected in parallel or series; they can present a wye and delta interconnection.

This section is devoted to obtaining the mathematical expressions that transform the mechanical impedances in delta interconnection (Figure 4a) to their equivalent mechanical impedances in wye interconnection (Figure 4b) and vice versa. The purpose of deducting this relation with impedances instead of particular mechanic elements is to present a general analysis.



(a) Delta configuration (b) Wye configuration

Figure 4: Wye and delta configurations (modified from [3])

To start the deduction, it is necessary to obtain the impedance Z_{12}, which is defined as the impedance between nodes 1 and 2. For the sake of simplification, Figure 4a can be seen as Figure 5. Z_C and Z_B are interconnected in series, so, their equivalent impedance is given by (3) (see Figure 6). As Z' and Z_A are in parallel, Z_{12} results in (4).

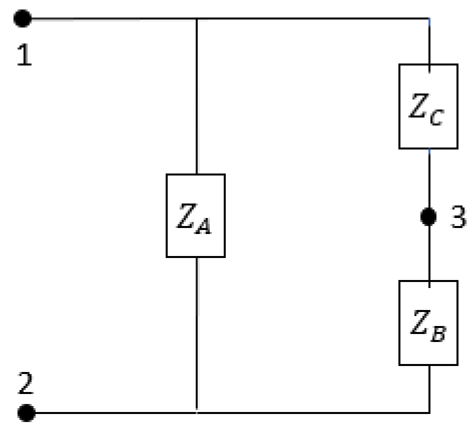


Figure 5: Delta configuration redrawn.

$$Z' = \frac{Z_B * Z_C}{Z_B + Z_C} \tag{3}$$

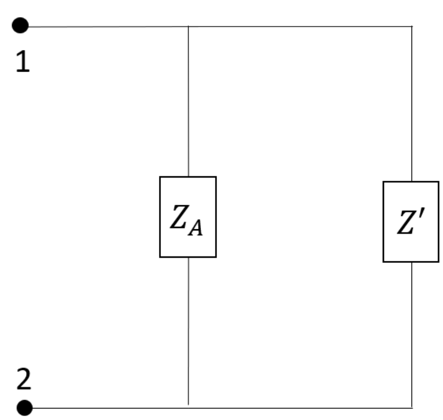


Figure 6: Simplified delta configuration.

$$Z_{12} = Z_A + \frac{Z_B * Z_C}{Z_B + Z_C} \tag{4}$$

On the other hand, the wye interconnection shown in Figure 4b can be seen as Figure 7, the impedance Z_{12} in the wye configuration is then in series with the Z_1 and Z_3 impedances, resulting 5.

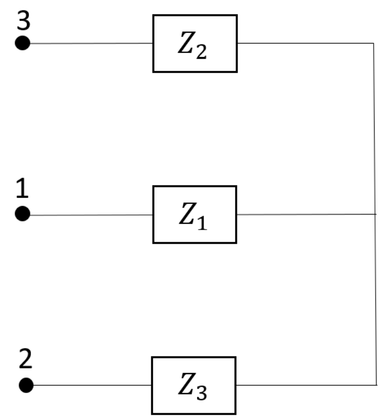


Figure 7: Wye configuration redrawn.

$$Z_{12} = \frac{Z_1 * Z_3}{Z_1 + Z_3} \tag{5}$$

Equation (6) is obtained by equating equations (4) and (5). Analogously, equation (7) is obtained for nodes 1 and 3, and (8) for nodes 2 and 3.

$$Z_A + \frac{Z_B * Z_C}{Z_B + Z_C} = \frac{Z_1 * Z_3}{Z_1 + Z_3} \tag{6}$$

$$Z_C + \frac{Z_A * Z_B}{Z_A + Z_B} = \frac{Z_1 * Z_2}{Z_1 + Z_2} \tag{7}$$

$$Z_B + \frac{Z_A * Z_C}{Z_A + Z_C} = \frac{Z_2 * Z_3}{Z_2 + Z_3} \tag{8}$$

Solving equations (6), (7) and (8) for Z_1, Z_2 and Z_3, equations (9) are obtained. Solving equations (6), (7) and (8) for Z_A, Z_B and Z_C, equations (10) are obtained.

$$Z_1 = \frac{Z_A * Z_B + Z_A * Z_C + Z_B * Z_C}{Z_B} \tag{9a}$$

$$Z_2 = \frac{Z_A * Z_B + Z_A * Z_C + Z_B * Z_C}{Z_A} \tag{9b}$$

$$Z_3 = \frac{Z_A * Z_B + Z_A * Z_C + Z_B * Z_C}{Z_C} \tag{9c}$$

$$Z_A = \frac{Z_1 * Z_3}{Z_1 + Z_2 + Z_3} \tag{10a}$$

$$Z_B = \frac{Z_2 * Z_3}{Z_1 + Z_2 + Z_3} \tag{10b}$$

$$Z_C = \frac{Z_1 * Z_2}{Z_1 + Z_2 + Z_3} \tag{10c}$$

The mathematical expressions that allow calculating the equivalent impedances in delta configuration from the impedances in wye configuration for electrical circuits are shown in equations (11a), (11b), and (11c). The mathematical expressions that allow calculating the equivalent impedances in wye configuration from the impedances in delta configuration for electrical circuits are shown in equations (12a), (12b), and (12c).

$$Z_A = \frac{Z_1 * Z_2 + Z_2 * Z_3 + Z_3 * Z_1}{Z_2} \tag{11a}$$

$$Z_B = \frac{Z_1 * Z_2 + Z_2 * Z_3 + Z_3 * Z_1}{Z_1} \tag{11b}$$

$$Z_C = \frac{Z_1 * Z_2 + Z_2 * Z_3 + Z_3 * Z_1}{Z_3} \tag{11c}$$

$$Z_1 = \frac{Z_A * Z_C}{Z_A + Z_B + Z_C} \tag{12a}$$

$$Z_2 = \frac{Z_B * Z_C}{Z_A + Z_B + Z_C} \tag{12b}$$

$$Z_3 = \frac{Z_A * Z_B}{Z_A + Z_B + Z_C} \tag{12c}$$

Note that equations (11) and (12) present the same structure that (9) and (10), then, it can be inferred that exists an analogy between wye and delta interconnection for mechanical and electrical systems.

C Drawing a mechanical circuit

The mechanical circuit is the representation of a mechanical system where the elements are connected by nodes and meshes. This representation allows identifying the elements that endure the same forces or displacements. This scheme will be a vital tool in obtaining the mathematical model of a mechanical system in an easier way than conventional methodologies. In this appendix, the steps to follow in order to draw a mechanical circuit from a mechanical system and some essential related concepts will be described.

A mechanical circuit contains main nodes and secondary nodes. The main nodes are those associated with the independent displacements in the system, and the secondary nodes are those that help to convert some main nodes from rotational movement to translational movement or vice versa, a secondary node is dependent of a main node associated.

If the system is subject to translational and rotational movement, the dimensions of the rotational mechanical elements help to know the relation between both. If there are two pulleys with radios r_1 and r_2 connected by a band with stiffness constant k, their representation in the mechanical circuit must be as in Figure 8.

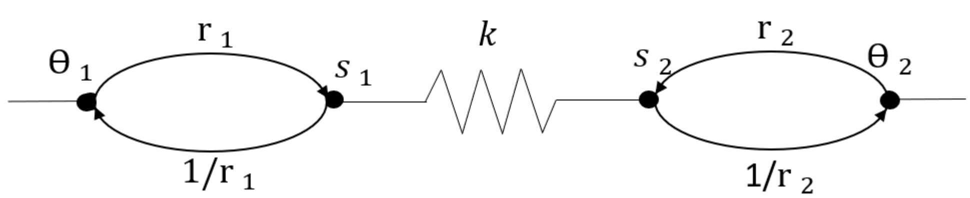


Figure 8: Polley representation in a mechanical circuit.

Figure 9 shows an schematic representation of two gears connected, with radios r_1 and r_2 in a mechanical circuit. Figure 10 shows an schematic representation of a lever with moment arms d_1 and d_2 between its extremes and the pivot.

For the representation of two gears connected, with radios r_1 and r_2 in the mechanical circuit see Figure 9. For a lever with moment arms d_1 and d_2

between its extremes and the pivot, see the Figure 10.

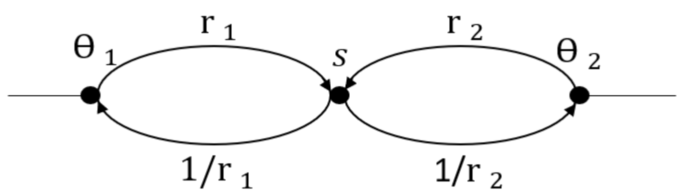


Figure 9: Gears representation in a mechanical circuit.

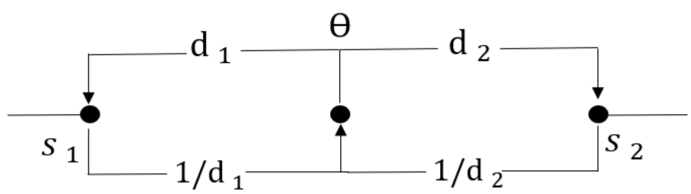


Figure 10: Lever representation in a mechanical circuit.

The symbol that will be used in the mechanical circuit to represent force sources is the analog to the symbol of a current source for electrical systems [13] shown in Figure 11.

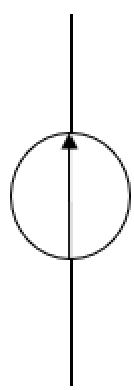


Figure 11: Force source

Each displacement and force source has a positive or a negative sense in the inertial reference frame. If the sense of a source coincides with the positive

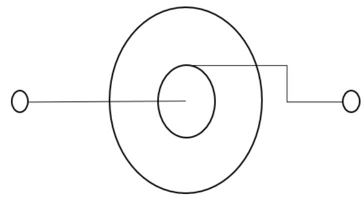


Figure 12: Inerter graphical representation (Modified from [4]).

direction of an axis in the inertial reference frame, it will be positive in the mechanical circuit representation. Otherwise, it will be negative. To draw a positive source in the mechanical circuit, the negative terminal of this source will be connected to the reference node. Otherwise, its positive terminal will be connected to the reference node [13]. In both cases, the remaining terminal will be connected to the mechanical element directly affected by the source.

The masses always have a terminal linked to the reference node. If the displacement of a mass has a negative sense, a minus sign beside its associated node in the mechanical system must be collocated.

On the other hand, an inerter does not need to be linked to the reference node. The representation of an inerter in a mechanical circuit for this work will by the symbol shown in Figure 12 [4].

One needs to identify the types of interconnections between the elements of the mechanical system. Whether the mechanical components endure the same displacement, they are in parallel interconnection, and whether they endure the same force acting on them, they are in series interconnection. To collocate the elements in the mechanical circuit, firstly, one must identify those that endure the same force or displacement with the elements connected to the sources of excitation, accommodate all the elements forming meshes or nodes among them respectively.

If there are mechanical elements in the mechanical system that are neither connected in parallel nor in series, it is considered that they are in delta or wye interconnection.

Within the relations among the mechanical elements consider that if there are pulleys, gears or levers, they need to be represented as shown in Figures (8), (9), and (10) respectively.

D Mathematical model of mechanical systems

The present section introduces the methodology to obtain the mathematical model of a mechanical system based on its representation in a mechanical circuit. The equations of movement will be obtained by applying the forces or displacements law to nodes or meshes, respectively.

First of all, if the mechanical circuit contains elements in delta or wye configuration, it is necessary to change its interconnection by using the equations (9) or (10). All the elements need to be interconnected in series or parallel. If there are two or more masses, springs, dampers or inerters in series or parallel, replace them by equivalent elements in order to simplify the circuit.

In order to decide whether to apply displacements or forces law to the mechanical circuit, the following items should be considered.

Since the nodal analysis works well with the forces through each mechanical component, if there are force sources between two meshes, nodal analysis is convenient as in [13].

Since the mesh analysis works with the displacement of each mechanical component, if there are displacement sources between two meshes, it is convenient to apply a mesh analysis as established in this document.

If excitement sources are found in a unique mesh, then a nodal or mesh analysis can be applied to obtain the mathematical model of the system. If there is a displacement source and the nodal analysis is chosen, then, the displacement of the node where the source is connected equals that of the source. For this node, it is not necessary to establish an equation due to the fact that the movement of the node is known.

If there is a force source in a unique mesh and the mesh analysis is chosen, then, the force in the mesh equals that of the source. For this mesh, it is not necessary to establish an equation due to the fact that force that would provoke the displacement of the elements in the mesh is known.

Although the nodal or mesh analysis can be utilized where there are excitement sources in an unique mesh, commonly, one of them implies more steps than the other. In order to reduce the number of equations, if the number of nodes in the mechanical circuit is higher than the number of meshes, the mesh analysis should be applied.

Similarly, if the number of meshes is higher than the number of nodes, nodal analysis should be applied. On the other hand, if one wants to obtain the equations of movement in terms of the forces that would generate displacements, one should opt for nodal analysis. If one wants to obtain the movement equations in terms of displacements, one should opt for a mesh analysis.

The mentioned conditions help to chose between the nodal or mesh analysis to obtain the equations of movement of a mechanical system. However, if there are displacement and force sources between meshes, then super nodes and super meshes might be used, as it will be explained below.

D.1 Super nodes and super meshes analysis

As mentioned above, when there are force and displacement sources in the middle of two meshes, the concept of super nodes and super meshes must be used.

To decide which of the two approaches should be chosen, the number of force and displacement sources need to be counted. If there are more force sources than displacement sources, the forces law using a super node might be used; then displacement law is applied in the super node. If there are more displacement sources than force sources, the displacement law is used in every mesh of the circuit, using the concept of super mesh. Then the force source is applied in the super mesh.

If the mechanical system contains rotational elements, an additional equation consisting of the sum of moments is required for the development of the mathematical model. In this case, the forces coming to each node, are multiplicated by the moment arm, relative to a point freely chosen.

The inertance is the mechanical impedance of an inerter. The inertance is measured in kilograms and is analogous to a mass. If a force is applied to the inerter, this force generates a proportional acceleration on it.

The following example shows how to cope with mechanical circuits where there is a force between two meshes. In this case, as was explained, a super mesh is used to obtain the mathematical model of the system.

Example 1. To obtain the mathematical model of the mechanical system shown in Figure 13, through the approach presented in this work, a circuit representation is required. The inertial reference frame is shown as a reference node in order to construct the mechanical circuit of the system shown in Figure 13. The mass is linked to the reference node, as well as the two excitation sources (see Figure 14).

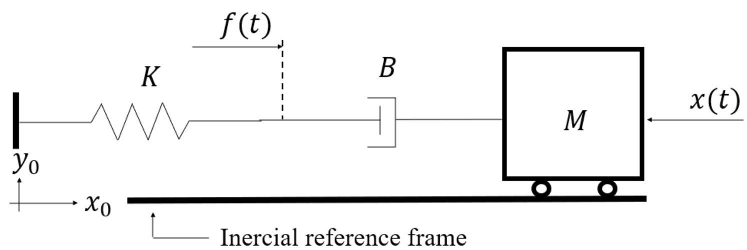


Figure 13: Mechanical system Example 1.

In the mechanical system, the force source has a positive sense, which is represented in the mechanical circuit connecting the arrow going out from the reference node. The displacement source has a negative sense in the mechanical system, so, to represent it in the mechanical circuit, its positive terminal is connected to the reference node.

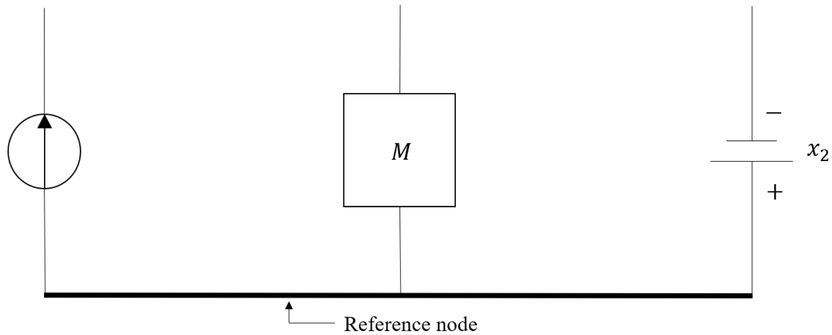


Figure 14: Mass and sources Example 1.

The position of the spring with the damper in the mechanical system is respected when the mechanical circuit is built. The springs are in contact with the reference frame and to the reference node, for the mechanical system and for the mechanical circuit, respectively. The damper is placed between the mass and the force source (see Figures 13 and 15).

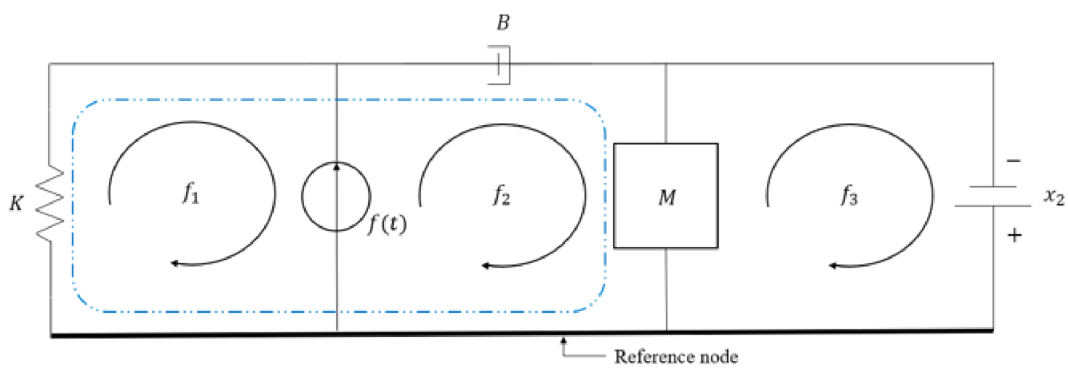


Figure 15: Mechanical circuit Example 1.

The use of a super mesh is illustrated in the modeling of the system given in Example 1. In the mechanical circuit of example 1, there are three meshes. Mesh 1 is considered as the mess formed with the spring and the source force. The second mesh includes the force source, the mass, and the damper. The mass and the displacement source form the third mesh.

There is a displacement source between mesh 1 and mesh 2, so, to make a mesh analysis, the use of a super mesh is needed. The forces that would generate the displacements in the elements of every mesh is drawn as a curved arrow with an aleatory sense. The super mesh is represented by color blue in Figure 15. Equation (13a) and equation (13b) are obtained by applying the displacement law for the super mesh and mesh 3, respectively. Then the forces law is applied to the super mesh, obtaining equation (13c). The behavior of the mechanical system illustrated in Figure 13 is described by the equations (13).

$$\frac{1}{k}f_1 + \frac{1}{BD}f_2 + \frac{1}{MD^2}(f_2 - f_3) = 0 \tag{13a}$$

$$\frac{1}{MD^2}(f_3 - f_2) = x_2 \tag{13b}$$

$$f(t) = f_2 - f_1 \tag{13c}$$