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SEQUENTIAL DECREASING STRONG SIZE PROPERTIES

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ABSTRACT. Let X be a continuum. The n-fold hyperspace $C_n(X)$, $n < \infty$, is the space of all nonempty closed subsets of X with at most n components. A topological property $\mathcal{P}$ is said to be a (an almost) sequential decreasing strong size property provided that if μ is a strong size map for $C_n(X)$, $\{t_j\}_{j=1}^{\infty}$ is a sequence in the interval $(t, 1)$ such that $\lim t_j = t \in [0, 1)$ ($t \in (0, 1)$) and each fiber $\mu^{-1}(t_j)$ has property $\mathcal{P}$, then so does $\mu^{-1}(t)$. In this paper we show that the following properties are sequential decreasing strong size properties: being a Kelley continuum, local connectedness, continuum chainability and, unicoherence. Also we prove that indecomposability is an almost sequential decreasing strong size property.

1. Introduction

In [12] and [13] F. Orozco-Zitli proved that atriodicity, containing no arc, irreducibility, indecomposability, being a Kelley continuum, local connectedness, continuum chainability and unicoherence are sequential decreasing Whitney properties. Sequential decreasing strong size properties are the natural generalization of sequential decreasing Whitney properties. We prove that being a Kelley continuum, local connectedness, continuum chainability and unicoherence are sequential decreasing strong size properties. Also we prove that indecomposibility is an almost sequential decreasing strong size property.

2. Preliminaries

Given a metric space (Z, d) and a subset B of Z. If $x \in Z$ and $\varepsilon > 0$, let $\mathcal{V}_{\varepsilon}^{d}(x) = \{y \in X : d(x, y) < \varepsilon\}$ and $N(\varepsilon, B) = \bigcup\{\mathcal{V}_{\varepsilon}^{d}(x) : x \in B\}$. We denote by $\mathrm{cl}(B)$ the closure of B in Z. Further, $\mathrm{diam}(B)$ will denote the diameter of B. A *continuum* is a nonempty compact, connected, metric space. A *subcontinuum* of a space Z is a continuum contained in Z.

The symbol $\mathbb{N}$ denotes the set of positive integers. Let X be a continuum. For each $n \in \mathbb{N}$, $C_n(X)$ denotes the hyperspace of all nonempty closed subsets of X with at most n components; $C_n(X)$ is called the *n-fold hyperspace* of X (thus, $C_1(X)$ is the classical hyperspace of all subcontinua of X and, as is customary, is denoted by $C(X)$ instead of $C_1(X)$). The symbol $F_n(X)$ denotes the *n-fold symmetric product* of a continuum X; that is, $F_n(X) = \{A \in C_n(X) : A \text{ has at most } n \text{ points}\}$. We topologize these sets with the Hausdorff metric H, defined as follows: $H(A, B) = \inf\{\varepsilon > 0 :$

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$A \subset N(\varepsilon, B)$ and $B \subset N(\varepsilon, A)\}$, (see [10: p. 1]). We denote by H^2 the corresponding Hausdorff metric for $C(C_n(X))$. An *order arc* in $C_n(X)$ is an arc $\alpha\colon [0,1] \to C_n(X)$ such that if $0 \leq s < t \leq 1$, then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$.

A *map* means a continuous function. A *size map* for $C_n(X)$ is a map $\omega\colon C_n(X) \to [0,1]$ such that $\omega(\{x\}) = 0$ for each $x \in X$ and $\omega(A) \leq \omega(B)$ if $A \subset B$ for each $A, B \in C_n(X)$. A *strong size map* for $C_n(X)$ is a map $\mu\colon C_n(X) \to [0,1]$ such that

(i) $\mu(A) = 0$ for each $A \in F_n(X)$,

(ii) if $A \subset B$, $A \neq B$ and $B \notin F_n(X)$, then $\mu(A) < \mu(B)$

(iii) $\mu(X) = 1$ (see [2: p. 956]).

By Theorem 2.10 of [2: p. 958], every strong size map is monotone. Each set of the form $\mu^{-1}(t)$ for any strong size map for $C_n(X)$ and any $t \in [0,1]$ is called a *strong size level* of $C_n(X)$.

Let X be a continuum and let μ be a strong size map for $C_n(X)$. Let $A \in C_n(X)$. If $t \in [0, \mu(A))$, let $C(A,t) = \{B \in \mu^{-1}(t) : B \subset A$ and each component of A intersects $B\}$. Also, if $t \in [\mu(A), 1)$, let $C_A^t = \{B \in \mu^{-1}(t) : A \subset B$ and each component of B intersects $A\}$. Notice that if $t \in [\mu(A), 1)$, then C_A^t is closed in $\mu^{-1}(t)$. If $t \in [0, \mu(A))$, then $C(A,t)$ is closed in $\mu^{-1}(t)$. Then, for each $t \in [0, \mu(A))$, $C(A,t)$ is a subcontinuum of $\mu^{-1}(t)$ (see [2: Theorem 2.14, p. 959]).

A topological property $\mathcal{P}$ is said to be a *sequential decreasing strong size property* provided that if μ is a strong size map for $C_n(X)$, $t \in [0,1)$, $\{t_j\}_{j\in\mathbb{N}}$ is a sequence into the interval $(t,1)$ such that $\lim t_j = t$ and each fiber $\mu^{-1}(t_j)$ has property $\mathcal{P}$, then so does $\mu^{-1}(t)$.

A topological property $\mathcal{P}$ is said to be an *almost sequential decreasing strong size property* provided that if μ is a strong size map for $C_n(X)$, $t \in (0,1)$, $\{t_j\}_{j\in\mathbb{N}}$ is a sequence into the interval $(t,1)$ such that $\lim t_j = t$ and each fiber $\mu^{-1}(t_j)$ has property $\mathcal{P}$, then so does $\mu^{-1}(t)$.

Let $\sigma\colon C(C_n(X)) \to C_n(X)$ be a function given by $\sigma(\mathcal{A}) = \bigcup\{A : A \in \mathcal{A}\}$, by [3: p. 23], σ is a map and, by [6: Lemma 7.2, p. 250]) it is well defined; it is clear that σ is onto. The map σ is called the union map.

A continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua. A continuum which is not decomposable is said to be *indecomposable*.

A continuum X is said to be *unicoherent* provided that whenever A and B are subcontinua of X such that $A \cup B = X$, then $A \cap B$ is connected.

A continuum X is called a *Kelley continuum* provided that given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $p, q \in X$ with $d(p,q) < \delta$ and $p \in A \in C(X)$, then there exists $B \in C(X)$ such that $q \in B$ and $H(A,B) < \varepsilon$.

A continuum X is *continuum chainable* if for each $\varepsilon > 0$ and each pair of points $p \neq q$ in X, there is a finite sequence of subcontinua $\{C_1, \ldots, C_r\}$ of X such that $\operatorname{diam}(C_i) < \epsilon$, $p \in C_1$, $q \in C_r$ and $C_i \cap C_{i+1} \neq \emptyset$ for every $i \leq r-1$.

Remark 2.1. It can easily be proved that a continuum X is a Kelley continuum if and only if for every point $p \in X$ and for each $\varepsilon > 0$, there exists $\delta > 0$ with the property that if $A \in C(X)$, $p \in A$ and $q \in \mathcal{V}_\delta^d(p)$, then there exists $B \in C(X)$ such that $q \in B$ and $H(A,B) < \varepsilon$.

3. Preliminary Results

Lemma 3.1. *Let X be a continuum. Let $\{A_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ be sequences of $C_n(X)$ such that $\lim A_k = A$ and $\lim B_k = B$. If and each component of B_k intersects A_k for each $k \in \mathbb{N}$, then each component of B intersects A.*

P r o o f. Let C be a component of B and let $x \in C$. Then there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $\lim x_k = x$ and $x_k \in B_k$ for each $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, let C_k be the component of B_k such that $x_k \in C_k$. Since $\{C_k\}_{k\in\mathbb{N}}$ is a sequence of elements of $C(X)$, by the compactness of $C(X)$ we may assume that $\{C_k\}_{k\in\mathbb{N}}$ converges to some element D of $C(X)$. Notice that $D \subset C$. Since $A_k \cap C_k \neq \emptyset$ for each $k \in \mathbb{N}$, $A \cap D \neq \emptyset$. Hence $A \cap C \neq \emptyset$. Therefore, every component of B intersects A. □

Lemma 3.2. *Let μ be a strong size map for $C_n(X)$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $A, B \in C_n(X)$ satisfy that each component of B intersects A, $A \subset N(\delta, B)$ and $|\mu(A) - \mu(B)| < \delta$, then $H(A, B) < \varepsilon$.*

P r o o f. Suppose that the lemma is false for some $\varepsilon > 0$. Then there are two sequences $\{A_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ in $C_n(X)$ such that, for each $m \in \mathbb{N}$, $A_m \subset N(\frac{1}{m}, B_m)$, each component of B_m intersects A_m, $|\mu(A_m) - \mu(B_m)| < \frac{1}{m}$ and $H(A_m, B_m) \geq \varepsilon$. We assume, without loss of generality, that $\lim A_k = A$ for some $A \in C_n(X)$ and $\lim B_k = B$ for some $B \in C_n(X)$. Notice that $A \subset B$. We will prove that $A = B$. If $B \in F_n(X)$, by Lemma 3.1, $A = B$. Now if $B \notin F_n(X)$, by the continuity of μ, $\mu(A) = \mu(B)$. Thus, $A = B$. Since $\lim B_k = B = A$ and $\lim A_k = A$, there exists $m \in \mathbb{N}$ such that $H(A_m, B_m) \leq H(A_m, A) + H(B_m, A) < \varepsilon$, a contradiction. □

Lemma 3.3. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0, 1)$. If $t \in (t_0, 1)$ and $\mathcal{A} \in C(\mu^{-1}(t))$, then $\bigcup\{C(A, t_0) : A \in \mathcal{A}\}$ is a subcontinuum of $\mu^{-1}(t_0)$.*

P r o o f. Let $\mathfrak{B} = \bigcup\{C(A, t_0) : A \in \mathcal{A}\}$. We will prove that $\mathfrak{B}$ is closed. Let $\{B_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$ such that $\lim B_k = B$ for some $B \in C_n(X)$. Then, there exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that, for each $k \in \mathbb{N}$, $B_k \in C(A_k, t_0)$. Since $\mathcal{A}$ is compact, we may assume that $\lim A_k = A$ for some $A \in \mathcal{A}$. Then, $B \subset A$ and $B \in \mu^{-1}(t_0)$. By Lemma 3.1, each component of A intersects B. Thus, $B \in C(A, t_0)$. Hence $B \in \mathfrak{B}$.
On the other hand, suppose that $\mathfrak{B}$ is not connected. Then, there are two nonempty disjoint closed subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathfrak{B}$ such that $\mathfrak{B} = \mathcal{L}_1 \cup \mathcal{L}_2$.

For each $i \in \{1, 2\}$, let $\mathcal{L}_i^* = \{A \in \mathcal{A} : C(A, t_0) \subset \mathcal{L}_i\}$. Notice that $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ are nonempty disjoint subsets of $\mathcal{A}$ and $\mathcal{L}_1^* \cup \mathcal{L}_2^* = \mathcal{A}$. Let $i \in \{1, 2\}$. In order to prove that $\mathcal{L}_i^*$ is closed, let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{L}_i^*$ converging to an element $A \in \mathcal{A}$. Since $\{C(A_k, t_0)\}_{k\in\mathbb{N}}$ is a sequence of elements of $C(\mu^{-1}(t_0))$. By compactness we may assume that the sequence $\{C(A_k, t_0)\}_{k\in\mathbb{N}}$ converges to an element $\mathcal{D} \in C(\mu^{-1}(t_0))$. Thus, since $\bigcup\limits_{k\in\mathbb{N}} C(A_k, t_0) \subset \mathcal{L}_i$ and $\mathcal{L}_i$ is closed, $\mathcal{D} \subset \mathcal{L}_i$. Now, we need to show that $\mathcal{D} \subset C(A, t_0)$. Let $B \in \mathcal{D}$. Then, there exists a sequence $\{B_k\}_{k\in\mathbb{N}}$ in $\mathfrak{B}$ such that, for each $k \in \mathbb{N}$, $B_k \in C(A_k, t_0)$ and $\lim B_k = B$. Then, $B \subset A$ and $B \in \mu^{-1}(t_0)$. By Lemma 3.1, $B \in C(A, t_0)$. We have shown that $\mathcal{D} \subset C(A, t_0)$. Thus, since $C(A, t_0)$ is connected, $C(A, t_0) \subset \mathcal{L}_i$. Hence $A \in \mathcal{L}_i^*$ and $\mathcal{L}_i^*$ is closed. Therefore, $\mathcal{A}$ is not connected, a contradiction.
This completes the proof that $\mathfrak{B}$ is a subcontinuum of $\mu^{-1}(t_0)$. □

The proof of the following lemma is similar to the one given for Lemma 3.2 of [8: p. 106] (see [10: Lemma 14.8.1, p. 406]).

Lemma 3.4. *Let μ be a strong size map for $C_n(X)$. If $A \in C_n(X)$ and $t \in (\mu(A), 1)$, then C_A^t is arcwise connected.*

Lemma 3.5. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0, 1)$. If $t \in (t_0, 1]$ and $\mathcal{A} \in C(\mu^{-1}(t_0))$, then $\bigcup\{C_A^t : A \in \mathcal{A}\}$ is a subcontinuum of $\mu^{-1}(t)$.*

P r o o f. Let $\mathcal{S} = \bigcup\{C_A^t : A \in \mathcal{A}\}$. Using similar ideas as in Lemma 3.3 we can prove that $\mathcal{S}$ is closed in $\mu^{-1}(t)$. Now suppose $\mathcal{S}$ is not connected. Then, there exist two nonempty disjoint closed subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{S}$ such that $\mathcal{S} = \mathcal{F}_1 \cup \mathcal{F}_2$. For each $i \in \{1, 2\}$, let $\mathcal{L}_i^* = \{A \in \mathcal{A} : C_A^t \subset \mathcal{F}_i\}$.

Notice that $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ are nonempty disjoint subsets of $\mathcal{A}$ and $\mathcal{L}_1^* \cup \mathcal{L}_2^* = \mathcal{A}$. Let $i \in \{1,2\}$. In order to prove that $\mathcal{L}_i^*$ is closed, we consider a sequence $\{A_k\}_{k\in\mathbb{N}}$ in $\mathcal{L}_i^*$ converging to an element $A \in \mathcal{A}$. Since $\{C^t_{A_k}\}_{k\in\mathbb{N}}$ is a sequence of elements of $C(\mu^{-1}(t))$. By compactness we may assume that $\{C^t_{A_k}\}_{k\in\mathbb{N}}$ converges to an element $\mathcal{D} \in C(\mu^{-1}(t))$. Thus, since $\bigcup_{k\in\mathbb{N}} C^t_{A_k} \subset \mathcal{F}_i$ and $\mathcal{F}_i$ is closed, $\mathcal{D} \subset \mathcal{F}_i$. Now, we need to show that $\mathcal{D} \subset C^t_A$. Let $B \in \mathcal{D}$. Then there exists a sequence $\{B_k\}_{k\in\mathbb{N}}$ in $\mathcal{S}$ such that, for each $k \in \mathbb{N}$, $B_k \in C^t_{A_k}$ and $\lim B_k = B$. Then $A \subset B$ and $B \in \mu^{-1}(t)$. By Lemma 3.1, $B \in C^t_A$. We have shown that $\mathcal{D} \subset C^t_A$. Thus, since C^t_A is connected (see Lemma 3.4), $C^t_A \subset \mathcal{F}_i$. Hence $A \in \mathcal{L}_i^*$ and $\mathcal{L}_i^*$ is closed. Therefore, $\mathcal{A}$ is not connected, a contradiction.
This completes the proof that $\mathcal{S}$ is a subcontinuum of $\mu^{-1}(t)$. □

For the following, it is known that if $\mathcal{A} \in C(C_n(X))$, then $\sigma(\mathcal{A}) \in C_n(X)$, see [7: Lemma 7.2].

Lemma 3.6. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $\mathcal{A}$ is a nondegenerate subcontinuum of $\mu^{-1}(t_0)$ and $t \in [t_0, \mu(\sigma(\mathcal{A})))$, then $X(\mathcal{A},t) = \{B \in \mu^{-1}(t)$: there exists a subcontinuum $\mathcal{B}$ of $\mathcal{A}$ such that $\sigma(\mathcal{B}) = B\}$ is a subcontinuum of $\mu^{-1}(t)$.*

Proof. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(s) = s - t_0$. Clearly, f is a homeomorphism. Define $\omega\colon C(\mathcal{A}) \to \mathbb{R}$ by $\omega(\mathfrak{B}) = f(\mu(\sigma(\mathfrak{B})))$. Notice that:

(1) ω is a map;

(2) $\omega(\{D\}) = 0$ for each $D \in \mathcal{A}$;

(3) if $\mathfrak{B}_1, \mathfrak{B}_2 \in C(\mathcal{A})$, with $\mathfrak{B}_1 \subset \mathfrak{B}_2$, then $\omega(\mathfrak{B}_1) \le \omega(\mathfrak{B}_2)$.

Thus, ω is a size map for $C(\mathcal{A})$. Hence for each $t \in [t_0, \mu(\sigma(\mathcal{A}))]$, $\omega^{-1}(f(t)) = \{\mathfrak{B} \in C(\mathcal{A}) : \mu(\sigma(\mathfrak{B})) = t\}$ is a subcontinuum of $C(\mathcal{A})$ (see [11: p. 243]). Since $X(\mathcal{A},t) = \sigma(\omega^{-1}(f(t)))$ and σ is continuous, $X(\mathcal{A},t)$ is a subcontinuum of $\mu^{-1}(t)$. □

Lemma 3.7. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $A \in \mu^{-1}(t_0)$ and $r \in (t_0,1)$, then there exists a subcontinuum $\mathcal{A}$ of $\mu^{-1}(t_0)$ such that $A \in \mathcal{A}$ and $\mu(\sigma(\mathcal{A})) = r$.*

Proof. Let $\alpha\colon [0,1] \to C(\mu^{-1}(t_0))$ be an order arc such that $\alpha(0) = \{A\}$ and $\alpha(1) = \mu^{-1}(t_0)$. Since the composition $\mu\circ\sigma\circ\alpha$ is continuous, and $\mu(\sigma(\alpha(0))) = t_0$ and $\mu(\sigma(\alpha(1))) = 1$, there exists $s \in (0,1)$ such that $\mu(\sigma(\alpha(s))) = r$. Note that $\alpha(s) \in C(C_n(X))$ because $C(\mu^{-1}(t_0)) \subset C(C_n(X))$. Clearly, $\mathcal{A} = \alpha(s)$ has the required properties, and the lemma is proved. □

Lemma 3.8. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $A, B \in \mu^{-1}(t_0)$ and $A \ne B$, then there exists $s \in (t_0,1)$ such that if $\mathcal{A}, \mathcal{B} \in C(\mu^{-1}(t_0))$, $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\mu(\sigma(\mathcal{A}))$, $\mu(\sigma(\mathcal{B})) \in (t_0,s)$, then $\sigma(\mathcal{A}) \ne \sigma(\mathcal{B})$.*

Proof. Let $a \in A \setminus B$ and let $\varepsilon > 0$ be such that $\mathcal{V}^d_\varepsilon(a) \cap B = \emptyset$. Let $\delta > 0$ be as in Lemma 3.2 for the number ε. Let $s = \min\{t_0 + \delta, 1\}$. Let $\mathcal{A}$ and $\mathcal{B}$ two subcontinua of $\mu^{-1}(t_0)$ such that $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\mu(\sigma(\mathcal{A}))$, $\mu(\sigma(\mathcal{B})) \in (t_0,s)$. Since $\mu(\sigma(\mathcal{B})) - \mu(B) < \delta$, $B \subset \sigma(\mathcal{B})$ and each component of $\sigma(\mathcal{B})$ intersects B (see [1: Lemma 3.1, p. 241]), by the choice of δ, $H(B,\sigma(\mathcal{B})) < \varepsilon$. Thus $\sigma(\mathcal{B}) \subset N(\epsilon, B)$. Therefore, $A \nsubseteq \sigma(\mathcal{B})$ and $\sigma(\mathcal{A}) \ne \sigma(\mathcal{B})$ □

4. Main Results

Theorem 4.1. *Local connectedness is a sequential decreasing strong size property.*

Proof. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $\{t_j\}_{j\in\mathbb{N}}$ is a sequence in $(t_0,1]$ converging to t_0 and each fiber $\mu^{-1}(t_j)$ is locally connected, we will prove that $\mu^{-1}(t_0)$ is locally connected. Let $\varepsilon > 0$. Let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon}{4}$. Let $t_J \in (t_0, t_0+\delta)$. Since $\mu^{-1}(t_J)$ is locally connected, by [14: 15.7, p. 23], there exists a finite set $\{\mathcal{A}_1,\dots,\mathcal{A}_m\}$ of subcontinua of $\mu^{-1}(t_J)$ such that $\operatorname{diam}(\mathcal{A}_i) < \frac{\varepsilon}{4}$ for each $i \le m$, and $\mu^{-1}(t_J) = \bigcup_{i=1}^{m} \mathcal{A}_i$. For each $i \in \{1,\dots,m\}$, define $\mathfrak{B}_i = \bigcup\{C(A,t_0) : A \in \mathcal{A}_i\}$.

Now we will prove that $\mu^{-1}(t_0) = \bigcup_{i=1}^{m} \mathfrak{B}_i$. Notice that by Lemma 3.3, for each $i \le m$, $\mathfrak{B}_i$ is a subcontinuum of $\mu^{-1}(t_0)$. On the other hand if $D \in \mu^{-1}(t_0)$, there exists an order arc $\alpha\colon [0,1] \to C_n(X)$ such that $\alpha(0) = D$ and $\alpha(1) = X$. Since $\mu\circ\alpha\colon [0,1] \to [0,1]$ is a mapping, there exists $s \in (0,1)$ such that $\mu(\alpha(s)) = t_J$. Notice that $\alpha(s) \in \mathcal{A}_i$ for some $i \in \{1,\dots,m\}$ and $\alpha(0) \subset \alpha(s)$ by definition of order arc. So, $D \in C(\alpha(s),t_0,n) \subset \mathcal{B}_i$. Thus $\mu^{-1}(t_0) = \bigcup_{i=1}^{m} \mathfrak{B}_i$. Finally we will show that $\operatorname{diam}(\mathfrak{B}_i) < \varepsilon$. Let $i \le m$. Consider $B \in \mathfrak{B}_i$ and $A \in \mathcal{A}_i$, such that $B \in C(A,t_0,n)$. Notice that $|\mu(A) - \mu(B)| < \delta$. So, by the choice of δ, $H(A,B) < \frac{\varepsilon}{4}$. Since $\operatorname{diam}(\mathcal{A}_i) < \frac{\varepsilon}{4}$, $H(M,B) < \frac{\varepsilon}{2}$ for each $M \in \mathcal{A}_i$. Therefore, $\operatorname{diam}(\mathfrak{B}_i) < \varepsilon$ and by [14: 15.7, p. 23], $\mu^{-1}(t_0)$ is locally connected. □

Theorem 4.2. *Continuum chainability is a sequential decreasing strong size property.*

Proof. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. Suppose that $\{t_j\}_{j\in\mathbb{N}} \subset (t_0,1]$ is a sequence which converges to t_0 and each fiber $\mu^{-1}(t_j)$ is continuum chainable.

In order to prove that $\mu^{-1}(t_0)$ is continuum chainable, let $A_1 \ne A_2 \in \mu^{-1}(t_0)$. Let $\varepsilon > 0$ and let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon}{4}$. For A_1 and A_2, let $s \in (t_0,1)$ be as in Lemma 3.8. Let $t_J \in (t_0, \min\{t_0+\delta, s\})$. By Lemma 3.7, for each $k \in \{1,2\}$, there exists $\mathcal{M}_k \in C(\mu^{-1}(t_0))$ such that $\mu(\sigma(\mathcal{M}_k)) = t_J$ and $A_k \in \mathcal{M}_k$. By [1: Lemma 3.1, p. 241], $A_k \in C(\sigma(\mathcal{M}_k),t_0)$ for each $k \in \{1,2\}$. By the choice of s, $\sigma(\mathcal{M}_1) \ne \sigma(\mathcal{M}_2)$. Since $\mu^{-1}(t_J)$ is continuum chainable, there exists a finite sequence $\{\mathcal{A}_1,\dots,\mathcal{A}_m\}$ of subcontinua of $\mu^{-1}(t_J)$ such that $\sigma(\mathcal{M}_1) \in \mathcal{A}_1$, $\sigma(\mathcal{M}_2) \in \mathcal{A}_m$, $\mathcal{A}_i \cap \mathcal{A}_{i+1} \ne \emptyset$ for each $i < m$ and $\operatorname{diam}(\mathcal{A}_i) < \frac{\varepsilon}{4}$, for each $i \le m$. By Lemma 3.3, $\mathfrak{B}_i = \bigcup\{C(D,t_0) : D \in \mathcal{A}_i\}$ is a subcontinuum of $\mu^{-1}(t_0)$, for each $i \in \{1,\dots,m\}$. Clearly, $A_1 \in \mathfrak{B}_1$, $A_2 \in \mathfrak{B}_m$ and $\mathfrak{B}_i \cap \mathfrak{B}_{i+1} \ne \emptyset$ for each $i < m$. Let $i \le m$. Now we show that $\operatorname{diam}(\mathfrak{B}_i) < \varepsilon$. Let $D \in \mathfrak{B}_i$. We consider $G \in \mathcal{A}_i$ such that $D \in C(G,t_0)$. Since $\mu(G) - \mu(D) < \delta$, by the choice of δ, $H(D,G) < \frac{\varepsilon}{4}$. So, since $\operatorname{diam}(\mathcal{A}_i) < \frac{\varepsilon}{4}$, $H(M,D) < \frac{\varepsilon}{2}$ for each $M \in \mathcal{A}_i$. Hence $\operatorname{diam}(\mathfrak{B}_i) < \varepsilon$. Since $\mathcal{A}_i \cap \mathcal{A}_{i+1} \ne \emptyset$ for each $i < m$, $\mathcal{B}_i \cap \mathcal{B}_{i+1} \ne \emptyset$ for each $i < m$. Therefore, $\mu^{-1}(t_0)$ is continuum chainable. □

Theorem 4.3. *The property of being a Kelley continuum is a sequential decreasing strong size property.*

Proof. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. Suppose that $\{t_j\}_{j\in\mathbb{N}} \subset (t_0,1]$ is a sequence converging t_0 and each fiber $\mu^{-1}(t_j)$ is a Kelley continuum.

We will prove that $\mu^{-1}(t_0)$ is a Kelley continuum. Suppose that the theorem is false for some $P \in \mu^{-1}(t_0)$ and some $\varepsilon > 0$. By Remark 2.1, there are two sequences $\{\mathcal{A}_m\}_{m\in\mathbb{N}} \subset C(\mu^{-1}(t_0))$ and $\{Q_m\}_{m\in\mathbb{N}} \subset \mu^{-1}(t_0)$ such that, for each $m \in \mathbb{N}$, $P \in \mathcal{A}_m$, $H(P,Q_m) < \frac{1}{m}$, and if $Q_m \in \mathcal{G} \in C(\mu^{-1}(t_0))$, $H^2(\mathcal{A}_m,\mathcal{G}) \ge \varepsilon$. Let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon}{12}$. Let $t_J \in (t_0,t_0+\delta)$. By Lemma 3.7, for each $m \in \mathbb{N}$, there exists $\mathcal{D}_m \in C(\mu^{-1}(t_0))$ such that $Q_m \in \mathcal{D}_m$ and $\mu(\sigma(\mathcal{D}_m)) = t_J$. We may assume that $\lim \mathcal{A}_m = \mathcal{A}$ and $\lim \mathcal{D}_m = \mathcal{D}$ for some $\mathcal{A},\mathcal{D} \in C(\mu^{-1}(t_0))$. Since $\lim Q_m = P \in \mathcal{A}$ and $Q_m \in \mathcal{D}_m$ for each $m \in \mathbb{N}$, we have $P \in \mathcal{D}$. Thus, $P \in \mathcal{D} \cap \mathcal{A}$ and therefore, $\mathcal{A} \cup \mathcal{D} \in C(\mu^{-1}(t_0))$.

We prove that $H^2(\mathcal{A},\mathcal{A}\cup\mathcal{D}) < \frac{\varepsilon}{6},$. To this end, it is enough to prove that $\mathrm{diam}(\mathcal{D}) < \frac{\varepsilon}{6}$ proving that $H(\sigma(\mathcal{D}),E) < \frac{\varepsilon}{12}$ for every $E \in \mathcal{D}$. Let $E \in \mathcal{D}$, by [1: Lemma 3.1, p. 241], $E \in C(\sigma(\mathcal{D}),t_0)$. Since $|\mu(\sigma(\mathcal{D})) - \mu(E)| = |t_J - t_0| < \delta$. By the choice of δ, $H(\sigma(\mathcal{D}),E) < \frac{\varepsilon}{12}$. Thus, $\mathrm{diam}(\mathcal{D}) < \frac{\varepsilon}{6}$. Hence $H^2(\mathcal{A},\mathcal{A}\cup\mathcal{D}) < \frac{\varepsilon}{6}$.

Notice that $\sigma(\mathcal{D}) \in X(\mathcal{A}\cup\mathcal{D},t_J)$. Since $\mu^{-1}(t_J)$ is a Kelley continuum, there exists $\eta > 0$ such that if $L \in \mu^{-1}(t_J)$ and $H(\sigma(\mathcal{D}),L) < \eta$, then there exists $\mathcal{B} \in C(\mu^{-1}(t_J))$ such that $L \in \mathcal{B}$ and $H^2(X(\mathcal{A}\cup\mathcal{D},t_J),\mathcal{B}) < \frac{\varepsilon}{12}$.

Let $M \geq 1$ be such that $H^2(\mathcal{A}_M,\mathcal{A}) < \frac{\varepsilon}{12}$ and $H^2(\mathcal{D},\mathcal{D}_M) < \eta$. Note that $H(\sigma(\mathcal{D}),\sigma(\mathcal{D}_M)) < \eta$. To prove this part, we take a point $x \in \sigma(\mathcal{D})$. By definition there exists $D \in \mathcal{D}$ such that $x \in D$, since $H^2(\mathcal{D},\mathcal{D}_M) < \eta$, there is $D_M \in \mathcal{D}_M$ such that $H(D,D_M) < \eta$. So, there exists $d_M \in D_M \subset \sigma(\mathcal{D}_M)$ such that $d(x,d) < \eta$. Therefore, $x \in N(\eta,\sigma(\mathcal{D}_M))$. Thus, $\sigma(\mathcal{D}) \subset N(\eta,\sigma(\mathcal{D}_M))$. Similarly we can prove that $\sigma(\mathcal{D}_M) \subset N(\eta,\sigma(\mathcal{D}))$. Then $H(\sigma(\mathcal{D}),\sigma(\mathcal{D}_M)) < \eta$. Let $\mathcal{B} \in C(\mu^{-1}(t_J))$ be such that $\sigma(\mathcal{D}_M) \in \mathcal{B}$ and $H^2(X(\mathcal{A}\cup\mathcal{D},t_J),\mathcal{B}) < \frac{\varepsilon}{12}$.

Let $\mathcal{G} = \bigcup\{C(G,t_0) : G \in \mathcal{B}\}$. By Lemma 3.3, $\mathcal{G} \in C(\mu^{-1}(t_0))$. Since $\sigma(\mathcal{D}_M) \in \mathcal{B}$ and $Q_M \in C(\sigma(\mathcal{D}_M),t_0)$, $Q_M \in \mathcal{G}$.

Now we prove that $H^2(\mathcal{A}\cup\mathcal{D},\mathcal{G}) < \frac{\varepsilon}{4}$. Let $R \in \mathcal{A}\cup\mathcal{D}$. Since $\mu(\sigma(\mathcal{A}\cup\mathcal{D})) \geq t_J$, by Lemma 3.7, there exists $\mathcal{L} \in C(\mathcal{A}\cup\mathcal{D})$ such that $R \in \mathcal{L}$ and $\mu(\sigma(\mathcal{L})) = t_J$. Notice that $R \in C(\sigma(\mathcal{L}),t_0)$ (see [1: Lemma 3.1, p. 241]). So, $\mu(R) = t_0$, Thus, $\mu(\sigma(\mathcal{L})) - \mu(R) = t_J - t_0 < \delta$ and by the choice of δ, $H(\sigma(\mathcal{L}),R) < \frac{\varepsilon}{12}$. Since $\sigma(\mathcal{L}) \in X(\mathcal{A}\cup\mathcal{D},t_J)$ and $H^2(X(\mathcal{A}\cup\mathcal{D},t_J),\mathcal{B}) < \frac{\varepsilon}{12}$, there exists $F' \in \mathcal{B}$ such that $H(\sigma(\mathcal{L}),F') < \frac{\varepsilon}{12}$. Let $S \in C(F',t_0)$. Since $F' \in \mathcal{B}$, $S \in \mathcal{G}$. Since $\mathcal{B} \in C(\mu^{-1}(t_J))$ and $\mu(F') - \mu(S) < \delta$, by the choice of δ, $H(S,F') < \frac{\varepsilon}{12}$. Thus, $H(R,S) < \frac{\varepsilon}{4}$. Hence $R \in N(\frac{\varepsilon}{4},\mathcal{G})$.
On the other hand, let $G \in \mathcal{B}$ and $D \in C(G,t_0)$. Since $\mu(G) - \mu(D) < \delta$, by the choice of δ, $H(G,D) < \frac{\varepsilon}{12}$. Since $\mathcal{B} \subset N(\frac{\varepsilon}{12},X(\mathcal{A}\cup\mathcal{D},t_J))$, there exists $F_1 \in X(\mathcal{A}\cup\mathcal{D},t_J)$ such that $H(G,F_1) < \frac{\varepsilon}{12}$. Since $F_1 \in X(\mathcal{A}\cup\mathcal{D},t_J)$, there exists $\mathcal{L} \in C(\mathcal{A}\cup\mathcal{D})$ such that $F_1 = \sigma(\mathcal{L})$ and $\mu(\sigma(\mathcal{L})) = t_J$. Let $E_1 \in \mathcal{L}$. By [1: Lemma 3.1, p. 241], $E_1 \in C(F,t_0)$. Since $\mu(F_1) - \mu(E_1) = t_J - t_0 < \delta$, by the choice of δ, $H(E_1,F_1) < \frac{\varepsilon}{12}$. So, $H(D,E_1) < \frac{\varepsilon}{4}$. Thus, $D \in N(\frac{\varepsilon}{4},\mathcal{A}\cup\mathcal{D})$. Hence $H^2(\mathcal{A}\cup\mathcal{D},\mathcal{G}) < \frac{\varepsilon}{4}$.

Therefore, $H^2(\mathcal{A}_M,\mathcal{G}) \leq H^2(\mathcal{A}_M,\mathcal{A}) + H^2(\mathcal{A},\mathcal{A}\cup\mathcal{D}) + H^2(\mathcal{A}\cup\mathcal{D},\mathcal{G}) < \frac{\varepsilon}{2}$, a contradiction. □

Theorem 4.4. *Unicoherence is a sequential decreasing strong size property.*

P r o o f. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. Suppose that $\{t_j\}_{j\in\mathbb{N}} \subset (t_0,1]$ is a sequence which converges to t_0 and each fiber $\mu^{-1}(t_j)$ is unicoherent.

Notice that $F_n(X)$ is unicoherent for each $n \geq 3$ (see [5: Theorem 8, p. 177]). So, since $\mu^{-1}(0) = F_n(X)$, $\mu^{-1}(0)$ is unicoherent for each $n \geq 3$.

In order to prove the other cases, we assume that $\mu^{-1}(t_0)$ is not unicoherent. Let $\mathcal{A}_1,\mathcal{A}_2 \in C(\mu^{-1}(t_0))$ be such that $\mu^{-1}(t_0) = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2$ is not connected. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two nonempty disjoint closed subsets of $\mu^{-1}(t_0)$ such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{F}_1 \cup \mathcal{F}_2$. Let $\varepsilon > 0$ be such that $N(\varepsilon,\mathcal{F}_1) \cap N(\varepsilon,\mathcal{F}_2) = \emptyset$.

For each $i \in \{1,2\}$, let $\mathcal{B}_i = \mathcal{A}_i \setminus (N(\varepsilon,\mathcal{F}_1) \cup N(\varepsilon,\mathcal{F}_2))$. Notice that $\mathcal{B}_1$ and $\mathcal{B}_2$ are nonempty disjoint closed subsets of $\mu^{-1}(t_0)$. Let $0 < \varepsilon_1 < \frac{\varepsilon}{8}$ be such that $N(\varepsilon_1,\mathcal{B}_1) \cap N(\varepsilon_1,\mathcal{B}_2) = \emptyset$. Let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon_1}{2}$. Let $t_J \in (t_0, t_0 + \delta)$. For each $i \in \{1,2\}$, let $\mathcal{C}_i = \bigcup\{C_D^{t_J} : D \in \mathcal{A}_i\}$.

We prove that $\mu^{-1}(t_J) = \mathcal{C}_1 \cup \mathcal{C}_2$. Let $P \in \mu^{-1}(t_J)$. Using order arcs, it can be shown that there exists $Q \in \mu^{-1}(t_0)$ such that $P \in C_Q^{t_J}$. So, $P \in \mathcal{C}_1 \cup \mathcal{C}_2$. On the other hand, by Lemma 3.3, $\mathcal{C}_1,\mathcal{C}_2 \in C(\mu^{-1}(t_J))$.

For each $i \in \{1,2\}$, let

$$\mathcal{G}_i = \{F \in \mu^{-1}(t_J) : \text{ there exists } A \in \mathrm{cl}(N(\frac{\varepsilon}{8},\mathcal{F}_i)) \text{ such that } F \in C_A^{t_J}\}.$$

We will show that $\mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{G}_1 \cup \mathcal{G}_2$. Let $D \in \mathcal{C}_1 \cap \mathcal{C}_2$. For each $i \in \{1,2\}$, let $A_i \in \mathcal{A}_i$ be such that $D \in C^{t_J}_{A_i}$. By the choice of δ, $H(A_1, A_2) \leq H(A_1, D) + H(A_2, D) < \varepsilon_1$. By the choice of ε_1, $\{A_1, A_2\} \cap (N(\frac{\varepsilon}{8}, \mathcal{F}_1) \cup N(\frac{\varepsilon}{8}, \mathcal{F}_2)) \neq \emptyset$. We assume, without loss of generality, that $A_1 \in N(\frac{\varepsilon}{8}, \mathcal{F}_1) \cup N(\frac{\varepsilon}{8}, \mathcal{F}_2)$. So, $D \in \mathcal{G}_1 \cup \mathcal{G}_2$.

In will prove that $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$. Let $G \in \mathcal{G}_1 \cap \mathcal{G}_2$. Then there exist $E_1 \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_1))$ and $E_2 \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_2))$ such that $G \in C^{t_J}_{E_1} \cap C^{t_J}_{E_2}$. By the choice of δ, $H(E_1, E_2) \leq H(E_1, G) + H(E_2, G) < \varepsilon_1$. For $i \in \{1,2\}$, let $F_i \in \mathcal{F}_i$ be such that $H(E_i, F_i) < \frac{\varepsilon}{4}$. Thus, $H(F_1, F_2) < \varepsilon$ which contradicts the choice of ε. We have shown that $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$.

Note that, given $F_i \in \mathcal{F}_i$, there exists $D_i \in \mu^{-1}(t_J)$ such that $D_i \in C^{t_J}_{F_i}$. Thus, $D_i \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_i$. We have shown that $\mathcal{G}_1$ and $\mathcal{G}_2$ are disjoint subsets of $\mu^{-1}(t_J)$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_1 \neq \emptyset \neq \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_2$.

Now, we prove that $\mathcal{G}_1$ is closed. Let $\{B_k\}_{k \in \mathbb{N}}$ be a sequence of $\mathcal{G}_1$ such that $\lim B_k = B$ for some $B \in \mu^{-1}(t_J)$. Notice that for each $k \in \mathbb{N}$, there exists $A_k \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_1))$ such that $B_k \in C^{t_J}_{A_k}$. By compactness we may assume that $\lim A_k = A$ for some $A \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_1))$. Since $B_k \in C^{t_J}_{A_k}$ for each $k \in \mathbb{N}$, $A \subset B$. By Lemma 3.1, $B \in C^{t_J}_{A}$. Hence $B \in \mathcal{G}_1$. Thus, $\mathcal{G}_1$ is closed. Similarly we can prove that $\mathcal{G}_2$ is closed.

Then $\mathcal{C}_1 \cap \mathcal{C}_2$ is disconnected. Therefore, $\mu^{-1}(t_J)$ is not unicoherent, a contradiction. □

It is known that for every $n > 1$, $F_n(X)$ is aposyndetic for every continuum X, and we know that every aposyndetic continuum is decomposable (see [4: Theorem 4, p. 289]). Thus, if X is a continuum and μ is a strong size map defined on $C_n(X)$, then $\mu^{-1}(0) = F_n(X)$. Hence $\mu^{-1}(0)$ is decomposable. Therefore, indecomposability is not a sequential decreasing strong size property.

Theorem 4.5. *Indecomposability is an almost sequential decreasing strong size property.*

P r o o f. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in (0,1)$. Suppose that $\{t_j\}_{j \in \mathbb{N}} \subset (t_0, 1]$ is a sequence which converges to t_0 and each fiber $\mu^{-1}(t_j)$ is indecomposable.

Suppose that there are two proper subcontinua $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mu^{-1}(t_0)$ such that $\mu^{-1}(t_0) = \mathcal{A}_1 \cup \mathcal{A}_2$. Let $A_1 \in \mathcal{A}_1 \setminus \mathcal{A}_2$ and $A_2 \in \mathcal{A}_2 \setminus \mathcal{A}_1$. Let $\varepsilon > 0$ be such that $\mathcal{V}^H_\varepsilon(A_1) \cap \mathcal{A}_2 = \emptyset = \mathcal{V}^H_\varepsilon(A_2) \cap \mathcal{A}_1$. Let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon}{2}$. Take $t_J \in (t_0, t_0 + \delta)$. For each $i \in \{1,2\}$, put $\mathcal{G}_i = \bigcup \{C^{t_J}_A : A \in \mathcal{A}_i\}$. We show that $\mu^{-1}(t_J) = \mathcal{G}_1 \cup \mathcal{G}_2$. Let $E \in \mu^{-1}(t_J)$. Using order arcs, it can be shown that there exists $F \in \mu^{-1}(t_0)$ such that $E \in C^{t_J}_F$. So, $E \in \mathcal{G}_1 \cup \mathcal{G}_2$. On the other hand, by Lemma 3.5, $\mathcal{G}_1, \mathcal{G}_2 \in C(\mu^{-1}(t_J))$.

Fix $G \in C^{t_J}_{A_1}$. If $G \in \mathcal{G}_2$, then $G \in C^{t_J}_R$ for some $R \in \mathcal{A}_2$. Since $\mu(G) - \mu(A_1)$, $\mu(G) - \mu(R) < \delta$, by the choice of δ, $H(R, G) < \frac{\varepsilon}{2}$ and $H(G, A_1) < \frac{\varepsilon}{2}$. So, $H(A_1, R) < \varepsilon$ which contradicts the choice of ε. Hence $\mathcal{G}_2 \neq \mu^{-1}(t_J)$. Similarly, $\mathcal{G}_1 \neq \mu^{-1}(t_J)$. Thus, $\mu^{-1}(t_J)$ is descomposable, a contradiction.
Therefore, $\mu^{-1}(t_0)$ is indecomposable. □

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