Generalized multistable structure via chaotic synchronization and preservation of scrolls


PII: S0016-0032(13)00252-4
DOI: http://dx.doi.org/10.1016/j.jfranklin.2013.06.025
Reference: FI1822

To appear in: Journal of the Franklin Institute

Received date: 18 September 2012
Revised date: 4 June 2013
Accepted date: 29 June 2013

Cite this article as: E. Jiménez-López, J.S. González Salas, L.J. Ontañón-García, E. Campos-Cantón, A.N. Pisarchik, Generalized multistable structure via chaotic synchronization and preservation of scrolls, Journal of the Franklin Institute, http://dx.doi.org/10.1016/j.jfranklin.2013.06.025

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting galley proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.
Generalized multistable structure via chaotic synchronization and preservation of scrolls

E. Jiménez-López\textsuperscript{a}, J.S. González Salas\textsuperscript{b}, L. J. Ontañón-García\textsuperscript{c}, E. Campos-Cantón\textsuperscript{d1} and A. N. Pisarchik \textsuperscript{e}

\textsuperscript{a} Departamento de Físico Matemáticas  
\textsuperscript{c} Instituto de Investigación en Comunicación Óptica  
Universidad Autónoma de San Luis Potosí  
Alvaro Obregón 64, Centro, 78000, S.L.P., México  
\textsuperscript{b} Academia de Matemáticas  
Universidad Politécnica de San Luis Potosí  
Urbano Villalón 500, 78369, S.L.P., México  
\textsuperscript{d} División de Matemáticas Aplicadas  
Instituto Potosino de Investigación Científica y Tecnológica  
Camino a la Presa San José 2055, Lomas 4 secc. 78216, S.L.P., México  
\textsuperscript{e} Centro de Investigaciones en Óptica  
Loma del Bosque 115, Lomas del Campestre, 37150 León, Gto., México

Abstract

Switched systems are capable of generating chaotic multi-scroll behavior in $\mathbb{R}^3$ by means of a control signal. This signal regulates an equilibrium position of the system and is defined according to the number of scrolls that is displayed by the attractor. Thus, if two systems are controlled by different signals, they exhibit a different number of scrolls. Multistability can be created by a pair of unidirectionally coupled unstable dissipative switched linear systems. A theoretical study of this phenomenon is performed with the jerky equations. Generalized synchronization is observed in numerical simulations of the master-slave system with different control signals. The

\textsuperscript{1}Corresponding author: eric.campos@ipicyt.edu.mx
proposed configuration preserves the number of scrolls and can possess an arbitrary large number of coexisting chaotic multi-scroll attractors.

**Keywords:** Multistability; chaotic multiscroll systems; generalized synchronization; piece-wise linear systems.

### 1. Introduction

Complex dissipative nonlinear systems often exhibit the coexistence of multiple stable equilibrium states for the same set of parameters. Some states may be chaotic, while others are periodic or fixed points. The coexistence of multiple attractors is one of the most exciting phenomena in nonlinear dynamics, which was observed in various systems, including electronic [1], lasers [2, 3], mechanical [4], chemical [5] and biological [6, 7] systems. Multistability, from the particular case of the coexistence of two attractors has been used for different applications, the most known is the generation of flip flops, widely used in memory storage devices. It has been recently shown that flip flops can be constructed using the base Chua’s circuit by means of standard interconnection between NAND gates [8] or using a bistability mode [9]. Multistability may be of particular relevance to biological systems, which switches between discrete states, generate oscillatory responses, or “remember” transitory stimuli [10].

Multistability often appears in coupled systems due to increasing complexity, when two or more systems join together. The emergence of multiple stable regions depends strongly on one (or the combination) of two factors: the coupling interconnection (linear, nonlinear, unidirectional or mutually coupled); the coupling strength. Dissipative weakly coupled systems can
present a very large number of coexisting attractors\[4\]. In other systems, the emergence of multistability often results in a loss of synchronization, e.g., in the formation of clusters in oscillators ensemble \[11, 12\] and in coupled logistic \[13\] and Hénon \[14, 15\] maps. Multistability in unidirectionally coupled identical systems is accompanied by different kinds of intermittent synchronization dependent on the coupling parameter \[16, 17, 18\].

In this work, we study the emergence of coexisting attractors in two multi-scroll chaotic systems coupled in a master-slave configuration, when the master and the slave systems display different number of scrolls. There are different approaches to obtain multi-scroll chaotic attractors. One of them is by switching a piece-wise linear (PWL) system \[19\] and controlling the stability of its equilibrium points. Other approaches imply the modification of the Chua’s system \[20, 21\] by replacing the nonlinear term with different nonlinear functions \[22, 23\]. Here, we consider unstable dissipative systems \[24\], a class of 3-D dynamical systems that presents multiple scrolls. This class of systems is constructed with a switching law to display various multi-scroll strange attractors, which result from a combination of several unstable “one-spiral” trajectories by means of changing the equilibria. Switching systems have been commonly used to switch only one system or the control signal. However there have been some approaches in which, instead of stabilizing the system the idea is to preserve or generate a specific dynamic \[25\]. Here we will show how multistability can be obtained using coupled switched systems.

The article is arranged as follows: Section 2 presents the theory that involves the unstable dissipative systems and the generation of multiple scrolls; Sec-
tion 3 describes the type of coupling used; Section 4 introduces the multimodal structures generated; the numerical results to prove the theory are depicted in Section 5; and conclusions are drawn in Section 6.

2. Multiscroll attractors by unstable dissipative systems

We consider the class of autonomous systems of linear differential equations given as follows:

\[ \dot{\chi} = A\chi + B, \quad (1) \]

where \( \chi = [x_1, x_2, x_3]^T \in \mathbb{R}^3 \) is the state variable, \( B = [\beta_1, \beta_2, \beta_3]^T \in \mathbb{R}^3 \) stands for a real vector, \( A = [\alpha_{ij}] \in \mathbb{R}^{3 \times 3} \) denotes a linear operator and the equilibrium point is located at \( \chi^* = -A^{-1}B \), which has a stable \( E^s \) and unstable \( E^u \) manifold. A Saddle equilibrium points, which connects stable and unstable manifolds play an important role in chaos creation, because they are responsible for successive stretching and folding processes. The stretching causes the system trajectories to exhibit sensitive dependence on initial conditions, whereas the folding creates a complicated microstructure [26]. The saddle points of a chaotic PWL system in \( \mathbb{R}^3 \) can be divided basic categories according to their eigenvalues \( \Lambda = \{\lambda_i, \lambda_j, \lambda_k\} \in \mathbb{C} \): (i) The saddle points that are stable in one direction only and unstable or oscillatory in the other two [24], i.e., it has one negative real eigenvalue \((Re\{\lambda_i\} < 0, Im\{\lambda_i\} = 0)\), and two complex conjugated eigenvalues \((Re\{\lambda_{j,k}\} > 0, Im\{\lambda_{j,k}\} \neq 0)\). (ii) The saddle points that are stable in two directions and unstable in only one, i.e., the dissipative components are oscillatory \((Im\{\lambda_{j,k}\} \neq 0 \text{ and } Re\{\lambda_{j,k}\} < 0)\), while the unstable component corresponds to a real positive eigenvalue \((Re\{\lambda_i\} > 0, Im\{\lambda_i\} = 0)\). If the system given
by Eq. (1) has a saddle equilibrium point responsible for unstable and stable manifolds and the sum of its eigenvalues is negative, then the system is called an *unstable dissipative system* (UDS). According to the above speculations, we define in the same way as [27] two different types of UDS, Type I and Type II as follows:

**Definition 2.1.** A system given by Eq. (1) in $\mathbb{R}^3$ with eigenvalues $\lambda_i$, $i = 1, 2, 3$, is said to be an UDS Type I, if $\sum_{i=1}^{3} \text{Re}\{\lambda_i\} < 0$ and one eigenvalue is positive real and the other two are complex conjugate with a negative real part.

**Definition 2.2.** A system given by Eq. (1) in $\mathbb{R}^3$ with eigenvalues $\lambda_i$, $i = 1, 2, 3$, is said to be an UDS Type II, if $\sum_{i=1}^{3} \text{Re}\{\lambda_i\} < 0$ and one eigenvalue is positive real and the other two are complex conjugate with a negative real part.

The above definitions imply that the UDS Type I is dissipative in one of its components, but unstable in the other two, which are oscillatory. The converse is the UDS Type II, which are dissipative and oscillatory in two of its components, but unstable in the other one. Since we are interested in the UDS Type I, the next corollary is important to note what kind of behaviors are possible in the system given by (1).

**Corollary 2.3.** Let the system (1) be a UDS Type I. Then, the system has a stable manifold $E^s = \text{span}\{\lambda_1\} \subset \mathbb{R}^3$ and another, unstable manifold $E^u = \text{span}\{\lambda_2, \lambda_3\} \subset \mathbb{R}^3$ and the following statements are true:

(a) All initial condition $\chi_0 \in \mathbb{R}^3 - E^s$ leads to a unstable orbit that goes to infinity.
(b) All initial condition $\chi_0 \in E^s$ leads to a stable orbit that settles down at $\chi^*$ and the system does not generate oscillations.

(c) The basin of attraction $D$ is $E^s \subset \mathbb{R}^3$.

Now, let us consider Luré-type systems of the form:

$$
\dot{\chi}(t) = A\chi(t) + Bu(t), \tag{2}
$$

$$
y(t) = G\chi(t), \tag{3}
$$

$$
u(t) = F(y(t)), \tag{4}
$$

where $G \in \mathbb{R}^3$, $F : \mathbb{R} \to \mathbb{R}$. In the control system, Eq. (2) represents a plant with control input signal $Bu(t)$, Eq. (3) is the system output (observable states), and Eq. (4) is the control signal which is a function of the output. Since the system (2) is unstable (because it present a saddle equilibrium point), it is necessary to design a control signal $u$ capable of stabilizing the system in a specific region. Different $F$ functions have been reported [28, 29]. Our goal is to design a function $F$ for Eq. (2) that generates a class of 3-D dynamical systems with chaotic oscillations, i.e., the flow $\Phi(\chi(t))$ of the system Eq. (2) is trapped in an attractor $A$ by means of modifying the function $F$. This class of systems can display various multi-scroll strange attractors as a result of the combination of several unstable “one-spiral” trajectories by means of $u$. In the other words, we are interesting in functions that can yield multi-scroll attractors constitute by a commuted signal, $u = S_i$, ($i = 1, \ldots, n$ and $n \geq 2$). Each signal $S_i$ has a domain $D_i \subset \mathbb{R}^3$, containing the equilibrium point $\chi_i^* = -A^{-1}BS_i$. Then, the function $F$ governs the dynamics of system (2) by changing the equilibria from $\chi_i^*$ to $\chi_j^*$, $i \neq j$, when
the flow $\Phi^t : D_i \rightarrow \mathbb{R}^3$ crosses from the $i-th$ to the $j-th$ domain. According to the above discussion it is possible to define a chaotic multi-scroll system based on UDS Type I as follows:

**Definition 2.4.** A system given by (2) in $\mathbb{R}^3$ and with equilibrium points $\chi_i^*$ ($i = 1, \ldots, n$ and $n \geq 2$) is said to be a chaotic multi-scroll system, if each $\chi_i^*$ contains oscillations around and the flow $\phi(\chi_0)$ generates an attractor $A \subset \mathbb{R}^3$.

The easiest chaotic system based on the UDS is defined with a step function $F$ commutated in two values, $S_1$ and $S_2$. Since the system Eq. (2) has two equilibria, a double scroll appears. The function $F$ is defined as a PWL function. Adding more PWL segments $S_i$ to the step function $F$, makes possible to obtain a number of scrolls proportional to the number of signals $S_i$. For simplicity, the function $F$ is given in terms of one state only, which defines the borders of domains as parallel planes orthogonal to one axis, that is achieved by defining $G = [1, 0, 0]$.

$$F(x_1(t)) = \begin{cases} S_1, & \text{if } x_1 \in D_1, \\ S_2, & \text{if } x_1 \in D_2, \\ \vdots, & \vdots \\ S_n & \text{if } x_1 \in D_n, \end{cases}$$

(5)

where $S_1, S_2, \ldots, S_n$ are constants, preserving the following order $S_1 < S_2 < \ldots < S_n$. In order to illustrate our approach, we consider the particular case of the linear ordinary differential equation (ODE) written in the equation of a jerky form as $\ddot{x} + \alpha_{33} \dot{x} + \alpha_{32} \dot{x} + \alpha_{31} x + \beta_3 = 0$, representing the state space of Eq. (1), where the matrix $A$ and the vector $B$ are given as follows:
\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\alpha_{31} & -\alpha_{32} & -\alpha_{33}
\end{pmatrix};
B = \begin{pmatrix}
0 \\
0 \\
\beta_3
\end{pmatrix},
\]  

(6)

with the coefficients \( \alpha_{31}, \alpha_{32}, \alpha_{33}, \beta_3 \in \mathbb{R} \). We should note that the \( A \) matrix is not restricted to the form derived from a jerky equation. The jerky form is just a convenient approach to build the matrices \( A \) and \( B \). The characteristic polynomial of matrix \( A \) given by Eq. (6) takes the following form:

\[
\lambda^3 + \alpha_{33}\lambda^2 + \alpha_{32}\lambda + \alpha_{31}.
\]  

(7)

The classical Descartes' rule of signs is a useful tool to state how many positive or negative roots can be expected from polynomial Eq. (7). Thus, since the system Eq. (6) is assumed dissipative, a direct implication yields

\[-\alpha_{33} = Tr(A) = \sum_{i=1}^{3} Re\{\lambda_i\} < 0.\]

Additionally, the system Eq. (6) has a saddle point at the equilibria as

\[-\alpha_{31} = \text{det}(A) < 0.\]

Moreover, \( \alpha_{32} > 0 \) in Eq. (7) is required to ensure that the real root of Eq. (7) is negative. Hence, under the above rationale, there are no positive real roots, because all coefficients in Eq. (7) are positive. Such implications give us the possibility that one of the three eigenvalues is a negative real value and the other two are complex with a real positive part.

3. Multimodal generalized synchronization

Consider a pair of unidirectionally coupled systems given as follows:
\[ \dot{\chi}^m = A\chi^m + Bu^m, \quad (8) \]
\[ \dot{\chi}^s = A\chi^s + Bu^s + C(\chi^m - \chi^s), \quad (9) \]

where the vectors \( \chi^m = (x^m_1, x^m_2, x^m_3)^T \in \mathbb{R}^3 \) and \( \chi^s = (x^s_1, x^s_2, x^s_3)^T \in \mathbb{R}^3 \) are the state variables of the master and slave systems, respectively, \( B = (0, 0, 1)^T \in \mathbb{R}^3 \), and \( C \in \mathbb{R}^{3 \times 3} \) is a constant matrix. In this work, we focus on the case when each system given by Eqs. (8) and (9) represent a different number of scrolls, i.e., \( \sharp S_m < \sharp S_s \), when \( C \) is the null matrix and \( u^m \neq u^s \). \( \sharp S_m \) and \( \sharp S_s \) stand for the number of scrolls that display the master and the slave system, respectively. When the systems are coupled, \( C \) is defined by just one real number as follows: \( (c_{22}) \neq 0 \), otherwise \( (c_{ij}) = 0, i, j = 1, 2, 3 \).

Using this kind of coupling, it is possible to explore generalized synchronization between a pair of different chaotic systems. The chaotic synchronization occurs whenever the master and the slave system flows are related by a function, that is:

\[ \lim_{t \to \infty} |\chi^m(t) - h(\chi^s(t))| = 0 \quad (10) \]

for every different pair of initial conditions \( \chi^m(0) = \chi^m_0 \) and \( \chi^s(0) = \chi^s_0 \) inside a ball of dissipation \( D \) (basin of attraction). For the case of generalized synchronization, the function \( h \) is difficult to find, but fortunately there are several approaches to determine generalized synchronization. One approach is by considering an auxiliary system \( \chi^a \) proposed in [30], which is a replica of the slave system. Using the auxiliary system, it is easy to detect generalized synchronization, meaning that
\[
\lim_{t \to \infty} |\chi_s(t) - \chi_a(t)| = 0. \tag{11}
\]

This is true, if there is no multistability and the system Eqs. (8) and (9) exhibit generalized synchronization. For the case of multistability, the basin of attraction is comprised by several basins of attraction \( \mathcal{D} = \bigcup_{k=1}^{v} \mathcal{D}_k \), where \( v \) represents a number of stable modes. Now, initial conditions for the slave and the auxiliary systems play an important role to detect general synchronization between the master and the slave systems.

\[
\lim_{t \to \infty} |\chi_s(t) - \chi_a(t)| = \begin{cases} 
0, & \text{for } \mathcal{D}_i(\chi_{s0}) = \mathcal{D}_j(\chi_{a0}), \\
d_{ij} > 0, & \text{for } \mathcal{D}_i(\chi_{s0}) \neq \mathcal{D}_j(\chi_{a0}),
\end{cases} \tag{12}
\]

where \( \chi_{s0} \in \mathcal{D}_i = \mathcal{D}_i(\chi_{s0}) \) and \( \chi_{a0} \in \mathcal{D}_j = \mathcal{D}_j(\chi_{a0}) \), \( i, j = 1 \ldots m \) and \( d \in \mathbb{R} \) is an arbitrary distance. There exist a \( c_0 \) value that guarantees generalized synchronization between the coupled systems given by Eqs. (8) and (9), \( c_0 \leq (c_22) \).

4. Generalized multistable structure

A multistable structure is created by the coupled system given by Eqs. (8) and (9), each in the jerky form Eq. (6) with \( \beta_3 = 1 \) in both the master and the slave systems. The coupled system is explicitly written as follows:
\[
\begin{align*}
\dot{x}_1^m &= x_2^m, \\
\dot{x}_2^m &= x_3^m, \\
\dot{x}_3^m &= -\alpha_{31}x_1^m - \alpha_{32}x_2^m - \alpha_{33}x_3^m + u^m(x_1^m), \\
\dot{x}_1^s &= x_2^s, \\
\dot{x}_2^s &= x_3^s + c_{22}(x_2^m - x_2^s), \\
\dot{x}_3^s &= -\alpha_{31}x_1^s - \alpha_{32}x_2^s - \alpha_{33}x_3^s + u^s(x_1^s). 
\end{align*}
\]  

(13)

**Theorem 4.1.** Let \( S_m = \{S_1^m, S_2^m, \ldots, S_{n_S}^m\} \) and \( S_s = \{S_1^s, S_2^s, \ldots, S_{n_S}^s\} \) be the sets conformed by the control signals \( u^m \) and \( u^s \), respectively. A given system in the form of Eq. (13) exhibits multistability if \( S_m^* \subset S_s \), and the number of multiple basins of attraction is \( n_s - n_m + 1 \), where \( n_s > n_m \) and \( S_m^* = \{S_1^m + \delta, S_2^m + \delta, \ldots, S_{n_S}^m + \delta\} \), where \( \delta \in \mathbb{R} \) is the displacement from one basin of attraction to another.

**Proof:** The solution of the fifth linear differential equation of the system (13) can be rewritten as follows:

\[
x_2^s(t) = e^{-c_{22}t} \int_0^t e^{c_{22}\tau}(c_{22}x_2^m(\tau) + x_3^s(\tau))d\tau + e^{-c_{22}t}x_2^s(0). 
\]  

(14)

Since we are interested in adiabatically stable solutions, \( \lim_{t \to \infty} e^{-c_{22}t}x_2^s(0) = 0 \). Therefore, we neglect the last term in Eq. (14). The next step is to integrate by parts Eq. (14), resulting in

\[
x_2^s(t) = x_2^m(t) + \frac{1}{c_{22}}x_3^s(t) - \frac{1}{c_{22}}e^{-c_{22}t} \int_0^t e^{c_{22}\tau} (c_{22}\dot{x}_2^m + \dot{x}_3^s) d\tau. 
\]  

(15)

Using the fact that \( \dot{x}_2^m = x_3^m \) and by applying integration by parts, the term with the integral in Eq. (15) can be developed as
\[
\frac{1}{c_{22}} e^{-c_{22} \tau} \int_0^t e^{c_{22} \tau} \left( c_{22} \dot{x}_2^m + \ddot{x}_2^s \right) d\tau = \frac{1}{c_{22}} x_3^m(t) + \frac{1}{c_{22}} e^{-c_{22} t} \int_0^t e^{c_{22} \tau} (\dot{x}_3^m - \ddot{x}_2^s) d\tau.
\]

From Eq. (15), we obtain

\[
x_2^s(t) = x_2^m(t) + \frac{1}{c_{22}} (x_3^s(t) - x_3^m(t)) + E(t),
\]

where \( E(t) = \frac{1}{c_{22}} e^{-c_{22} t} \int_0^t e^{c_{22} \tau} (\dot{x}_3^m - \ddot{x}_2^s) d\tau \). Assuming that \( i) c_{22} \gg 1 \), without loss of generality \( E(t) \approx 0 \), \( ii) c_{22} \gg |x_3^s(t) - x_3^m(t)| \) gives as a consequence \( (x_3^m(t) - x_3^s(t))/c_{22} \approx 0 \). Under the above assumptions and after a transient time, Eq. (16) becomes \( x_2^m(t) \approx x_2^s(t) \). Now, if \( \rho_1(t) = x_1^m(t) - x_1^s(t) \), \( \rho_2(t) = x_2^m(t) - x_2^s(t) \), \( \rho_3(t) = x_3^m(t) - x_3^s(t) \), then we obtain

\[
\begin{align*}
\dot{\rho}_1(t) &= \dot{x}_1^m(t) - \dot{x}_1^s(t) = x_2^m(t) - x_2^s(t) \approx 0, \\
\dot{\rho}_2(t) &= \dot{x}_2^m(t) - \dot{x}_2^s(t) = x_3^m(t) - x_3^s(t) \approx 0, \\
\dot{\rho}_3(t) &= \dot{x}_3^m(t) - \dot{x}_3^s(t),
\end{align*}
\]

that results in the difference between the first states of the master and the slave systems as \( x_1^m(t) - x_1^s(t) = K(x_1^m(0), x_2^m(0)) \).

If we consider that \( x_1^s \approx x_1^m + K(x_1^m(0), x_2^m(0)) \), \( x_2^s \approx x_2^m \), then for the sixth state of the system (13), we obtain

\[
\dot{x}_3^s \approx -\alpha_{31} x_1^s - \alpha_{32} x_2^m - \alpha_{33} x_3^s + u_s(x_1^s).
\]

By calculating the fixed points of the last equation, after transients we obtain \( x_1^m = (1/\alpha_{31})u^m(x_1^m) \), \( x_2^m = 0 \), \( x_3^s = 0 \) resulting in
\[ u^s(x^s_1) = u^m(x^m_1) + 1.5K(x^m_1(0), x^m_2(0)). \] (18)

The last expression establishes that the \( u^s \) signal applied to the slave system of Eq. (13) is given as a function of the \( u^m \) signal applied to the master system of Eq. (13), but shifted by a constant, which depends on the initial conditions \( x_1(0) \) and \( x_2(0) \). Besides, if we substitute \( u^s \) from Eq. (18) into Eq. (17), the fixed points of the slave system are \((2/3u^m(x^m_1) + K(x_1(0), x_2(0)), 0)\), i.e., the slave has the same fixed points as the master, but shifted by a constant \( K \) dependent on the initial conditions. The \( u^m \) signals are restricted to take consecutive values of \( u^s \), i.e., if \( \mathcal{G}_m = \{S^m_1, S^m_2, S^m_3\} \), \( \mathcal{G}_s = \{S^s_1, S^s_2, S^s_3, S^s_4, S^s_5\} \), and \( \mathcal{G}_m \subset \mathcal{G}_s \) then the possibilities are \( \{S^m_1 = S^s_1, S^m_2 = S^s_2, S^m_3 = S^s_3\} \), \( \{S^m_1 = S^s_2, S^m_2 = S^s_3, S^m_3 = S^s_4\} \), and \( \{S^m_1 = S^s_3, S^m_2 = S^s_4, S^m_3 = S^s_5\} \), but never \( \{S^m_1 = S^s_1, S^m_2 = S^s_3, S^m_3 = S^s_4\} \). Therefore, the multistable behavior is justified.

5. Numerical Results

In order to illustrate our approach, we consider the master system from Eq. (8) given by the \( A \) matrix and the \( B \) vector as follows

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.5 & -1 & -1 \end{pmatrix}; B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\] (19)

and the control signal given by a real function \( u^m_i : \mathbb{R} \to \mathbb{R} \) (henceforth \( i \) denotes a number of scrolls that the orbit follows). The master system is
Figure 1: Projection of chaotic multi-scroll attractors onto \((x_1^m, x_2^m)\) plane, generated by different control signal \(u_i\): (a) Eq. (20), (b) Eq. (21), (c) Eq. (22), and (d) Eq. (23).

based on UDS type I because the \(A\) matrix has three eigenvalues: one negative real eigenvalue \(\lambda_1 = -1.2041\), and the others are complex eigenvalues with positive real part \(\lambda_2 = 0.1020 + i1.1111\), \(\lambda_3 = 0.1020 - i1.1111\). The system is dissipative due to \(\sum_{i=1}^{3} Re\{\lambda_i\} < 0\). Thus, a double-scroll system can be given by Eqs. (8) and (19), and \(u_2^m\) as follows

\[
u_2^m = \begin{cases} 
0.9, & \text{if } x_1 \geq 0.3; \\
0, & \text{otherwise}.
\end{cases}
\] (20)

The equilibrium points of the system given by Eqs. (8), (19), and (20) are \(\chi_1 = (0.6, 0, 0)\) and \(\chi_2^* = (0, 0, 0)\). Figure 1 (a) depicts the projection of the double-scroll chaotic attractor onto the \((x_1^m, x_2^m)\) plane. It is possible to generate a triple-scroll attractor by modifying the \(u_i\) function. Thus, \(u_3^m\) is
given as follows

\[
u^m_3 = \begin{cases} 
0.9, & \text{if } x_1 \geq 0.3, \\
0, & \text{if } -0.3 < x_1 < 0.3, \\
-0.9, & \text{if } x_1 \leq -0.3.
\end{cases}
\] (21)

Notice that \( \chi^*_3 = -\chi^*_1 \). This issue is intentionally defined to illustrate the symmetry in scrolls. Figure 1 (b) shows the projection of the triple-scroll chaotic attractor onto the \((x^m_1, x^m_2)\) plane by using Eqs. (8), (19), and (21). This approach can be extended in order to generate systems with any number of scrolls, i.e., quadruple and quintuple scroll chaotic attractors are produced by defining \( u_4 \) and \( u_5 \) in accordance with the following features:

\[
u^m_4 = \begin{cases} 
1.8, & \text{if } x_1 \geq 0.9, \\
0.9, & \text{if } 0.3 \leq x_1 < 0.9, \\
0, & \text{if } -0.3 < x_1 < 0.3, \\
-0.9, & \text{if } x_1 \leq -0.3.
\end{cases}
\] (22)

\[
u^m_5 = \begin{cases} 
1.8, & \text{if } x_1 \geq 0.9, \\
0.9, & \text{if } 0.3 \leq x_1 < 0.9, \\
0, & \text{if } -0.3 < x_1 < 0.3, \\
-0.9, & \text{if } -0.9 < x_1 \leq -0.3, \\
-1.8, & \text{if } x_1 \leq -0.9.
\end{cases}
\] (23)

The functions given by Eqs. (22) and (23) introduce other equilibrium points located at \( \chi^*_4 = (1.2, 0, 0) \) and \( \chi^*_5 = (-1.2, 0, 0) \), respectively. Figures 1 (c) and (d) show the projection of the quadruple-scroll and quintuple-scroll chaotic attractors onto the \((x^m_1, x^m_2)\) plane, respectively.
Figure 2: Projection of a chaotic multi-scroll attractor onto \((x^s_1, x^s_2)\) plane, generated with the control signal given by Eq. (24).

Now, we consider the coupled system (13) given by the master system from Eqs. (8), (19), and (21), and the slave system from Eqs. (9) and (19) with

\[
\begin{align*}
    u^s_6 = \begin{cases}
        2.7, & \text{if } x_1 \geq 1.5, \\
        1.8, & \text{if } 0.9 \leq x_1 < 1.5, \\
        0.9, & \text{if } 0.3 < x_1 < 0.9, \\
        0, & \text{if } -0.3 < x_1 \leq 0.3, \\
        -0.9, & \text{if } -0.9 < x_1 \leq -0.3, \\
        -1.8, & \text{if } x_1 \leq -0.9.
\end{cases}
\end{align*}
\] (24)

Using the algorithm and based on the definition described by Wolf et al. [31], the maximum Lyapunov exponent for the system (19) is 0.10185, showing that the system is chaotic.

Figure 1 (b) shows the projection onto the \((x^m_1, x^m_2)\) plane of the mas-
Figure 3: The $d$ distance via $c_{22}$ coupling parameter for master and slave systems given by (a) $u_3^m$ and $u_6^s$, (b) $u_4^m$ and $u_6^s$, and (c) $u_5^m$ and $u_6^s$, respectively.

The master system, and Figure 2 shows the projection of the slave system onto the $(x_1^s, x_2^s)$ plane when $c_{22} = 0$. The slave system displays six scrolls with equilibria points $\chi^*_j$ for $j = 1, 2, 3, 4, 5$, given previously, and $\chi^*_6 = (1.8, 0, 0)$. The sets $\mathcal{G}_m = \{-0.9, 0, 0.9\}$ and $\mathcal{G}_s = \{-1.8, -0.9, 0, 0.9, 1.8, 2.7\}$ have the cardinality $\#\mathcal{G}_m = 3$ and $\#\mathcal{G}_s = 6$, respectively. According to Theorem 4.1, this system presents multistability due to $\mathcal{G}_m^* \subset \mathcal{G}_s$, with $\delta = 0$ and there are $\#\mathcal{G}_s - \#\mathcal{G}_{m+1} = 4$ basins of attraction. By choosing arbitrary initial conditions for the slave system $\chi^a(0)$ and its auxiliary system $\chi^a(0)$ inside the ball of dissipation $\mathcal{D}$, we calculate the euclidian distance between their orbits in order to check the limit given by Eq. (11), where

$$d = \frac{1}{N} \sum_{j=1}^{N} \sqrt{(x_1^s(j) - x_1^a(j))^2 + (x_2^s(j) - x_2^a(j))^2 + (x_3^s(j) - x_3^a(j))^2}.$$  

Figure 3 shows the $d$ distance via the $c_{22}$ coupling parameter when the master sys-
Figure 4: Generalized synchronization between master and slave systems. Projections of a chaotic multi-scroll attractor onto (a) \((x_s^1, x_s^2)\) plane of slave system; and (b) \((x_s^1, x_s^3)\) plane of slave and auxiliary systems.

The master and slave system are given by \(u^m_3\) and \(u^s_6\), respectively. Figure 3 (a) shows \(d = 0\) (marked in blue) for \(c_{22} \in (4.5, 26)\) when the initial conditions belong to the same basin of attraction, and three distances \(d \neq 0\) (marked in red, brown and black) when the initial conditions belong to a different basin of attraction, proving that there is multistability. Figure 3 (b) shows the \(d\) distance when the master and slave systems are given by \(u^m_4\) and \(u^s_6\), respectively, thus exhibiting three basins of attraction. Finally, Fig. 3 (c) displays the \(d\) distance for \(u^m_5\) and \(u^s_6\), thus giving two basins of attraction.

Figure 4 (a) shows the projection of the unidirectionally coupled slave system given by Eqs. (13) and (19) onto the \((x_s^1, x_s^2)\) plane for \(c_{22} = 5\). It can be seen that the number of scrolls of the master system is preserved in the slave system because \(u^m_3\) and \(u^s_6\) are given by Eqs. (21) and (24), respectively. Figure 4 (b) shows the projection of the slave and auxiliary...
systems onto the \((x_1^s, x_2^s)\) plane when their initial conditions belong to the same basin of attraction.

In the master-slave system, the number of scrolls of the master system is preserved in the slave system independently of the number of scrolls in the slave system. To illustrate this, the control signal \(u_{10}^s\) is changed as follows

\[
u_{10}^s = \begin{cases} 
8.1, & \text{if } x \geq 6.7, \\
7.2, & \text{if } 5.1 \leq x_1 < 6.7, \\
6.3, & \text{if } 4.5 \leq x_1 < 5.1, \\
5.4, & \text{if } 3.9 \leq x_1 \leq 4.5, \\
4.5, & \text{if } 3.3 \leq x_1 < 3.9, \\
3.6, & \text{if } 2.1 \leq x_1 < 2.7, \\
2.7, & \text{if } 1.5 \leq x_1 < 2.1, \\
1.8, & \text{if } 0.9 \leq x_1 < 1.5, \\
0.9, & \text{if } 0.3 \leq x_1 < 0.9, \\
0, & \text{if } x_1 < 0.3.
\end{cases}
\] (25)

Now, the slave attractor presents ten scrolls when it is uncoupled and its equilibrium points are given by \(\chi_i = (6/10i, 0, 0)\), \(i = 0, 1, \cdots, 9\). The projection of the slave system onto the \((x_1^s, x_2^s)\) plane is shown in Figure 5. The number of scroll is numerated from left to right. The sets \(S_m = \{-0.9, 0, 0.9\}\) and \(S_s = \{0, 0.9, 1.8, 2.7, 3.6, 4.5, 5.4, 6.3, 7.2, 8.1\}\) have the follow cardinality \(\#S_m = 3\) and \(\#S_s = 10\), respectively. According to Theorem 4.1, this system presents multistability due to \(S^*_m \subset G_s\), with \(\delta = 0.9\) and there are \(\#S_s - \#S_{m+1} = 8\) basins of attraction.

Figure 6 shows the projections of eight attractors of the slave system onto the \((x_1^s, x_2^s)\) plane obtained by changing the initial conditions only. The
Figure 5: Projection of a chaotic multi-scroll attractor of the slave system onto \((x_1^s, x_2^s)\) plane using the \(u_{s10}\) control signal Eq. (25).

The projection of the slave system trajectory for \(c_{22} = 0\) is shown by very tenuous gray lines, resulting from the slave system before being forced. We can see that the slave system oscillates preserving the number of scrolls of the master system, giving rise to multistability phenomena.

Table 1 describes the attractors in Figure 6. The first column indicates the initial conditions of the slave system, the second column shows the number of scroll, and the third column indicates the attractor color.

Since we are interested in the study of multistability, we vary the initial conditions of the slave system on the \((x_1^s, x_2^s)\) plane, while fixing the initial conditions of the master system at \(\chi^m(0) = (1, 1, 1)\). Figure 7 shows eight basins of attraction onto a region of the plane \(x_3^s = 0\) \((x_1^s \in [-1, 7] \text{ and } x_2^s \in [-2, 2])\) due to multistability of the coupled system (13) using \(u_{m3}\) and \(u_{s10}\).

The basin of attraction marked with blue dots corresponds to the attractor that covers the scrolls 1, 2 and 3 (see Figure 6). The basin of attraction
Figure 6: Projection of multiple coexisting chaotic multi-scroll attractors onto $(x_1, x_2)$ plane.
<table>
<thead>
<tr>
<th>Initial Conditions</th>
<th>Scrolls covered</th>
<th>Color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1, 0.1, 0)</td>
<td>1, 2, 3</td>
<td>Blue</td>
</tr>
<tr>
<td>(0.6, 0.6, 0)</td>
<td>2, 3, 4</td>
<td>Red</td>
</tr>
<tr>
<td>(1.5, 1.5, 0)</td>
<td>3, 4, 5</td>
<td>Green</td>
</tr>
<tr>
<td>(2, 2, 0)</td>
<td>4, 5, 6</td>
<td>Purple</td>
</tr>
<tr>
<td>(2.5, 2.5, 0)</td>
<td>5, 6, 7</td>
<td>Yellow</td>
</tr>
<tr>
<td>(3, 3, 0)</td>
<td>6, 7, 8</td>
<td>Brown</td>
</tr>
<tr>
<td>(3.6, 3.6, 0)</td>
<td>7, 8, 9</td>
<td>Cyan</td>
</tr>
<tr>
<td>(5, 5, 0)</td>
<td>8, 9, 10</td>
<td>Pink</td>
</tr>
</tbody>
</table>

Table 1: List of initial conditions, number of scrolls and colors of the orbits in Figure 6 for the resulting slave system.

Figure 7: Eight basins of attraction onto plane $x_3 = 0$ given by the couple system (13) using $u_2^{10}$ and $u_1^{10}$.
marked with red dots corresponds to the attractor that covers the scrolls 2, 3 and 4. The colors of the basins of attraction in Figure 7 are related with the colors of the attractors shown in Figure 6.

6. Conclusions

In this work, we have evaluated a mechanism for constructing generalized multistable structures in a pair of unidirectionally coupled systems. Particularly, we have demonstrated the coexistence of multiple attractors in a chaotic multi-scroll system composed by two unidirectionally coupled unstable dissipative subsystems, when they present generalized synchronization. To illustrate this approach, the parameters of the jerky equation were tuned to satisfy the definition of UDS type I. The control signal used to generate a desired number of scrolls, was given by a step function. The chaotic behavior of the proposed system is justified by the Lyapunov exponent analysis. We have shown that the number of scroll of the master system is preserved in the slave attractor under our approach when generalized synchronization exits. That is, the master system determines the number of scroll in the slave system by means of synchronization, which induces preservation on the number of scrolls in the slave system. In a particular case, when the number of scrolls in the master system is less than the number of scroll in the slave system, the master-slave configuration results in multiple basins of attraction for the slave. If two identical systems (slave and auxiliary) are synchronized by a third different system (master), which is coupled unidirectionally with the first two, we say that there exists generalized synchronization between the master and the slave systems. But due to the coexistence of multiple attrac-
tors, the two identical systems (slave and auxiliary) can be led to the same or different basin of attraction depending on their initial conditions. This phenomenon is called multimodal generalized synchronization meaning that a functional relation between the master and the slave subsystems exists.

This approach may be further extended to generate: 1) coexistence of chaotic attractors by UDS Type II subsystems, and 2) coexistence of chaotic attractors by using other control signals $u$.

Acknowledgments

E. J. L. thanks IPICYT for the hospitality during his sojourn in DMAp-IPICYT. E. J. L. and L. J. O. G. are doctoral fellows of CONACYT (Mexico) in the Graduate Program on Applied Science at IICO-UASLP. E. C. C. acknowledges CONACYT for the financial support through project No. 181002.


