

Exact solutions of a quartic potential

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We find that the analytical solutions to quantum system with a quartic potential $V(x) = ax^2 + bx^4$ (arbitrary a and $b > 0$ are real numbers) are given by the triconfluent Heun functions $H_T(\alpha, \beta, \gamma; z)$. The properties of the wave functions, which are strongly relevant for the potential parameters a and b , are illustrated. It is shown that the wave functions are shrunk to the origin for a given b when the potential parameter a increases, while the wave peak of wave functions is concaved to the origin when the negative potential parameter $|a|$ increases or parameter b decreases for a given negative potential parameter a . The minimum value of the double well case ($a < 0$) is given by $V_{\min} = -a^2/(4b)$ at $x = \pm\sqrt{|a|/2b}$.

Keywords: Exact solution; quartic potential; triconfluent Heun function.

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1. Introduction

As we know, the exact solutions of quantum systems have become an important topic in quantum mechanics. For example, we often choose the hydrogen atom

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and harmonic oscillator as two typical examples to explain quantum phenomena in almost all quantum mechanics textbooks.^{1,2} Up to now, some popular methods are used to solve these soluble quantum systems. The first is the functional analysis method, i.e. one solves the second-order differential equations and obtains their solutions expressed by the well-known special functions.³ The second is called the algebraic method, which is relevant for the SUSYQM⁴ and essentially connected to the factorization method.⁵ The third is named the exact quantization rule method⁶⁻⁸ and further developed to the proper quantization rule method,⁹ which shows more beauty and symmetry than the exact quantization rule. Almost all soluble potentials solved by these methods mentioned above belong to single-well potentials except for the double-well potentials,¹⁰⁻²¹ some of which are found that their solutions can be written as the confluent Heun function through some variable transformations.²²⁻²⁹ In this work, we are going to study the well-known quartic potential³⁰

$$V(x) = ax^2 + bx^4, \quad b > 0, \quad (1)$$

where a and b are two real number parameters. Since this potential includes the harmonic oscillator part, there are many different methods used to study it. The typical methods include the perturbation method,³¹ path integral method,³² algebraic approach,^{33,34} quasilinearization method,^{35,36} Bethe Ansatz method,³⁷⁻⁴⁰ asymptotic series method,⁴¹ generalized Bohr-Sommerfeld rule⁴² and others. The solutions obtained by all these methods are not exact except for approximate results. In this work, we will show that the Schrödinger equation with this potential can be solved exactly, that is, its solutions are given by triconfluent Heun function $H_T(\alpha, \beta, \gamma; z)$, which plays the similar role to the confluent Heun functions appeared in recent study.⁴³⁻⁴⁷

After studying this potential (1), we find that the minimum value of the potential $V_{\min} = -a^2/(4b)$ at $x = \pm\sqrt{|a|/\sqrt{2b}}$ when the potential parameter a is taken as negative. We note that the parameter $a < 0$ corresponds to a double well potential. Obviously, the depth of the potential well depends on two parameters a and b . In Fig. 1 we plot this potential as a function of the variable x with different values a and b , respectively. The purpose of this work is to study the solutions of the potential (1) and to see whether its solutions can be written as one of the special functions. After studying this potential, we find that its solutions can be written as the triconfluent Heun functions $H_T(\alpha, \beta, \gamma; z)$, but the energy levels are involved inside the parameter α of this triconfluent Heun function. To our best knowledge, this exact solution has never been reported in the literature. Undoubtedly, this result will enrich the exactly soluble quantum system in quantum mechanics.

This paper is organized as follows. In Sec. 2, we show how to obtain the solutions of the Schrödinger equation with this potential. This can be realized by transforming the Schrödinger equation into a triconfluent Heun differential equation through taking some variable transformations. In Sec. 3, the fundamental properties of the wave functions are studied and illustrated graphically for two different cases,

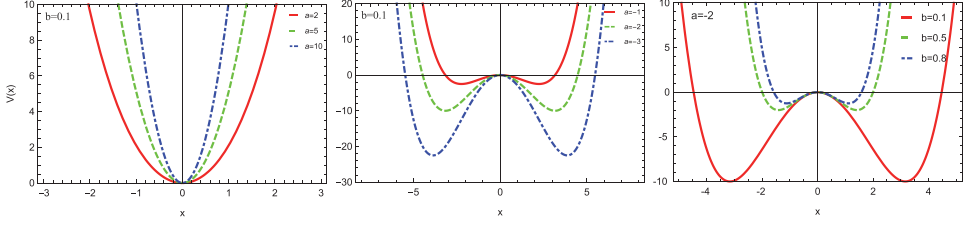


Fig. 1. (Color online) Plots of potential as a function of the variable x for different values of potential parameters a and b .

i.e. the positive and negative potential parameter a corresponding to the single- and double-well potentials. The energy levels for different parameter values a and b are calculated numerically. We summarize our results and give some useful remarks in Sec. 4.

2. Exact Solutions

Let us consider the one-dimensional Schrödinger equation ($\hbar = \mu = 1$),

$$-\frac{1}{2} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = \epsilon\psi(x). \quad (2)$$

By substituting potential (1) into (2), we have

$$\frac{d^2}{dx^2} \psi(x) + 2\{\epsilon - (ax^2 + bx^4)\}\psi(x) = 0. \quad (3)$$

We first consider the behavior of the wave functions at infinity. When the variable x tends to infinity, we may take the wave functions of the form

$$\psi(x) = F(x)e^{-\frac{\sqrt{2x(3a+2bx^2)}}{6\sqrt{b}}}, \quad (4)$$

and then substitute this into above Eq. (3). Thus, we are able to obtain the following differential equation

$$\frac{(a^2 - 4\sqrt{2b^3}x + 4b\epsilon)}{2b} F(x) - \frac{\sqrt{2}(a + 2bx^2)}{\sqrt{b}} F'(x) + F''(x) = 0. \quad (5)$$

Now, our task is how to solve this differential equation. To this end, by taking a new variable

$$z = \sqrt[6]{\frac{8b}{9}}x, \quad (6)$$

and then substituting this into Eq. (5), we obtain the following differential equation about $F(z)$:

$$\left(\frac{1}{4} \sqrt[3]{\frac{9}{b^4}} (4b\epsilon + a^2) - 3z \right) F(z) - \left(a \sqrt[3]{\frac{3}{b^2}} + 3z^2 \right) F'(z) + F''(z) = 0. \quad (7)$$

Compared this with the triconfluent Heun differential equation in the simplest uniform form²⁶

$$\frac{d^2 H(z)}{dz^2} - (\gamma + 3z^2) \frac{dH(z)}{dz} - [(3 - \beta)z - \alpha]H(z) = 0, \tag{8}$$

we have

$$\alpha = \frac{1}{4} \sqrt[3]{\frac{9}{b^4}} (4b\epsilon + a^2), \quad \beta = 0, \quad \gamma = a \sqrt[3]{\frac{3}{b^2}}. \tag{9}$$

Thus, the solutions of this system are given by

$$\psi(x) = e^{-\frac{\sqrt{2x(3a+2bx^2)}}{6\sqrt{b}}} H_T \left(\frac{1}{4} \sqrt[3]{\frac{9}{b^4}} (4b\epsilon + a^2), 0, a \sqrt[3]{\frac{3}{b^2}}; \sqrt[6]{\frac{8b}{9}} x \right). \tag{10}$$

This implies that the parameter α is related to energy levels. The wave functions given by $H_T(\alpha, \beta, \gamma; z)$ seem to be analytical, but the key issue is how to first get the energy levels. On the other hand, there exists another solution, i.e.

$$\psi(x) = e^{\frac{\sqrt{2x(3a+2bx^2)}}{6\sqrt{b}}} H_T \left(\frac{1}{4} \sqrt[3]{\frac{9}{b^4}} (4b\epsilon + a^2), 0, a \sqrt[3]{\frac{3}{b^2}}; -\sqrt[6]{\frac{8b}{9}} x \right), \tag{11}$$

which can be also obtained easily from the first solution (10) because of the symmetry $x \rightarrow -x$. Therefore, how to obtain the eigenvalues becomes a challenging task, but this can be solved numerically in another way (see Appendix A).

Before ending this section, let us consider the particular case $\beta = 0$, as shown in Ref. 26 (see pp. 262–265) to obtain a few interesting polynomial solutions of the low-lying states. According to Proposition 3.1.1: *The space $V_{\alpha, \beta, \gamma}$ contains a one-dimensional subspace of polynomials of degree d if and only if the two following conditions are fulfilled: (a) $\beta = 3(d + 1)$ and (b) $\prod_{d+1}(\alpha, \gamma) = 0$, where $\prod_{d+1}(\alpha, \beta)$ is the determinant of dimension $d + 1$ given by*

$$\begin{vmatrix} \alpha & -\gamma & 2 \cdot 1 & 0 & \dots & & & & & \\ 3d & \alpha & -2\gamma & 3 \cdot 2 & \dots & & & & & \\ 0 & 3(d-1) & \alpha & -3\gamma & \dots & & & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 3 \cdot 3 & \alpha & -(d-1)\gamma & d(d-1) & \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \cdot 2 & \alpha & -d\gamma & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \cdot 1 & \alpha & \end{vmatrix}. \tag{12}$$

This determinant is obtained by the following recurrent relation after substituting the polynomial $P(x) = \sum_{k=0}^d \lambda_k x^k$ into (8)

$$(\beta - 3k)\lambda_{k-1} + \alpha\lambda_k - \gamma(k+1)\lambda_{k+1} + (k+1)(k+2)\lambda_{k+2} = 0. \tag{13}$$

Based on the second condition $\prod_{d+1}(\alpha, \gamma) = 0$, some polynomials for small d can be obtained by

$$\begin{aligned}
 d = 0, \quad \beta = 3, \quad \alpha = 0, \quad P_0(x) &= 1, \\
 d = 1, \quad \beta = 6, \quad \alpha^2 + 3\gamma = 0, \quad P_1(x) &= x - \frac{\alpha}{3}, \\
 d = 2, \quad \beta = 9, \quad \alpha^3 + 12\alpha\gamma + 36 = 0, \\
 P_2(x) &= x^2 - \frac{\alpha}{3}x + \frac{\alpha^2}{36} - \frac{1}{\alpha}, \\
 d = 3, \quad \beta = 12, \quad \alpha^4 + 30\alpha^2\gamma + 216\alpha + 81\gamma^2 = 0, \\
 P_3(x) &= x^3 - \frac{\alpha}{3}x^2 + \left(\frac{\gamma}{2} + \frac{\alpha^2}{18}\right)x - \frac{\alpha^3}{162} - \frac{7\alpha\gamma}{54} - \frac{2}{3}.
 \end{aligned} \tag{14}$$

In principle, we can calculate the energy spectra from the constraint relations among the parameters α and γ given above. For example, when $\alpha = 0$, one has $\epsilon = -a^2/4b$. Other cases can also be calculated in this way. Unfortunately, we recognize that the energy levels calculated from this are not reliable and different from the numerical results. In addition, it would fail to get them since the first condition is violated due to the particular case $\beta = 0$. On the other hand, as suggested in Ref. 26, the change of variable $x \rightarrow x^2$ transforms Eq. (3) to one with lower class, but such a transformation will change the essence of the problem, that is, the interval of variable $x^2 \in [0, \infty)$ but not the original $x \in (-\infty, \infty)$.

3. Fundamental Properties

Now, let us study some basic properties of the wave functions (see Figs. 2–4). We consider this system from a few different cases, i.e. varying a and b , respectively. We show in Fig. 2 that the wave functions are shrunk to the origin for a given $b = 0.1$ when the potential parameter $a > 0$ increases. This leads to the increasing

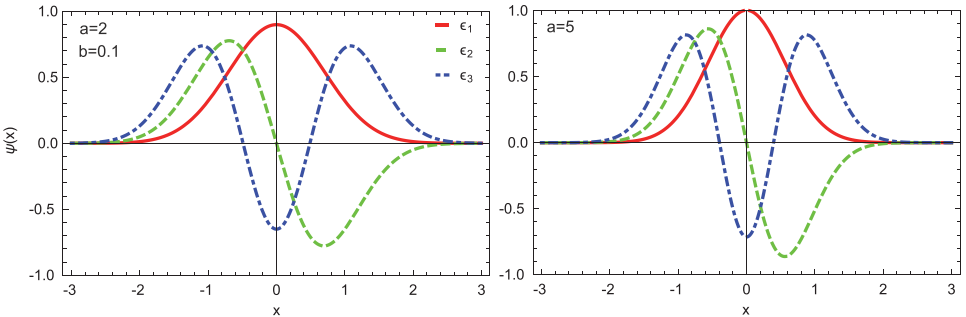


Fig. 2. (Color online) Plot of the wave functions as a function of the variable x . The potential parameter a is taken as $g = 2, 5$ and $b = 0.1$.

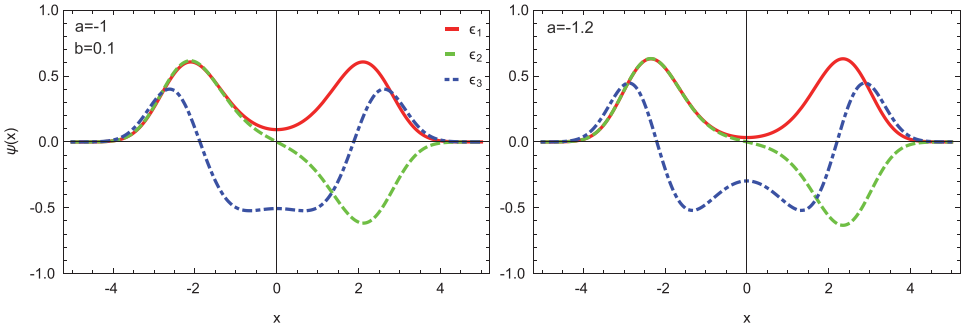


Fig. 3. (Color online) The wave functions plotted as a function of the variable x . The potential parameter a is taken as $a = -1.0, -1.2$, respectively, and $b = 0.1$.

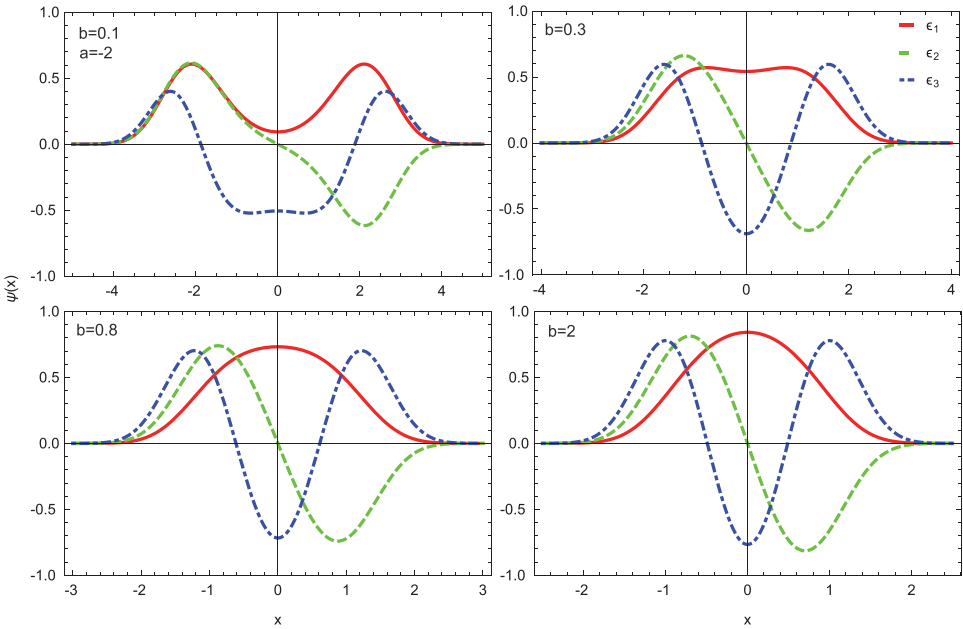


Fig. 4. (Color online) The wave functions plotted as a function of the variable x . The potential parameter b is taken as $b = 0.1, 0.3, 0.8, 2$, respectively, and $a = -2$.

amplitude of the wave functions. We illustrate in Fig. 3 the wave functions for a given $b = 0.1$ as the negative potential parameter $|a|$ increases. As we see, when the negative $|a|$ increases, the depth of the potential well is increasing quickly. The wave peaks of the wave functions have been concave to the origin for a given $b = 0.1$ due to the increasing negative potential parameter $|a|$. Figure 4 shows the plots of the wave functions for a given $a = -2$ and the increasing potential parameter b . We find that the wave peaks of the wave functions also concave to the origin when the

Table 1. Spectra of the Schrödinger equation with potential (1) and $b = 0.1$.

a	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7
1	0.74091	2.28229	3.93281	5.67596	7.50059	9.39853	11.3634
2	1.01800	3.0881	5.2236	7.41942	9.67142	11.9761	14.3305
3	1.23696	3.73457	6.27835	8.8661	11.4959	14.1659	16.8746
4	1.42345	4.28843	7.18898	10.1239	13.0921	16.0927	19.1246
5	1.58856	4.78029	8.00088	11.2496	14.5259	17.8290	21.1584
6	1.73825	5.22700	8.74002	12.2768	15.8370	19.4202	23.0259
7	1.87615	5.63900	9.42277	13.2271	17.0518	20.8965	24.7609

Table 2. Spectra of the Schrödinger equation with potential (1) and $b = 0.1$.

a	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7
-0.2	0.169454	0.776816	1.73556	2.85281	4.10599	5.47113	6.93325
-0.4	-0.0233079	0.376458	1.25673	2.26058	3.41756	4.69279	6.07034
-0.6	-0.321555	-0.119237	0.766008	1.62736	2.69033	3.87472	5.1676
-0.8	-0.816627	-0.748655	0.307123	0.950269	1.93194	3.01908	4.22634
-1.0	-1.56544	-1.55035	-0.145754	0.212126	1.17148	2.13126	3.25163
-1.2	-2.55197	-2.54951	-0.755072	-0.631357	0.479852	1.21552	2.25739
-1.4	-3.75555	-1.66542	-1.63989	-0.140693	0.256997	1.28571	2.27762

Table 3. Spectra of the Schrödinger equation with potential (1) and $a = -2$.

b	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7
0.5	-0.855175	-0.623961	0.970719	2.30647	4.06072	6.01864	8.16826
1.0	-0.130419	0.661429	2.54379	4.66539	7.12623	9.84367	12.7834
1.5	0.132469	1.30245	3.52739	6.10636	9.04386	12.2678	15.7393
2.0	0.29447	1.74452	4.25596	7.18218	10.4903	14.1087	17.9954
2.5	0.413446	2.08845	4.84364	8.05633	11.6728	15.6199	19.8532
3.0	0.508547	2.37327	5.34128	8.80073	12.6839	16.9157	21.4495
3.5	0.588407	2.61829	5.77596	9.4538	13.5737	18.0584	22.8592

potential parameter b decreases. We list the energy levels $\epsilon_i (i \in [1, 7])$ in Tables 1–3 for the various a and b .

4. Conclusions

In this work, we have studied the quantum system with a quartic potential and found that its exact solutions can be expressed as the triconfluent Heun functions $\psi(x) = e^{\mp \frac{\sqrt{2}x(3a+2bx^2)}{6\sqrt{b}}} H_T(\alpha, \beta, \gamma; \pm \sqrt{\frac{8b}{9}}x)$ by transforming the original differential equation into a triconfluent Heun differential equation. The fact that the energy levels are involved inside the parameter α means that we have to calculate the eigenvalues numerically. The properties of the wave functions depending on the potential parameters a and b have been illustrated graphically. We find that the wave functions are shrunk to the origin for a given b when the potential parameter a increases,

while the wave peak of wave functions is concaved to the origin when the negative potential parameter $|a|$ increases or the parameter b decreases for a given negative potential parameter a .

Before ending this work, we would like to give a few useful remarks on this potential. First, since the quartic potential belongs to the single- and double-well potentials for the positive and negative potential parameter a , respectively, perhaps the quartic potential could be used to treat the vibrations of real diatomic molecules. The present results may provide useful information for predicting the enthalpy values, entropy values and Gibbs-free energies for some diatomic molecule substances.^{48–52} Second, the traditional method to solve this class of second-order differential Heun equation is the series method, as done in Refs. 18 and 26. This method, also named as the Frobenius series method, requires the series are expanded near the origin or singularity. The energy levels obtained by this way have some constraints on the potential parameters, as summarized in our recent book.⁴⁰ Moreover, the previous study about this potential presents such a limit, i.e. $b \ll a$. However, our present case is concerned with *arbitrary* potential parameter a and the positive $b > 0$. Third, the study in this work could be extended to three-dimensional case, in particular, the S wave case ($l = 0$). However, the interval of the position domain becomes x (or r) $\in [0, \infty)$, the node numbers cannot be increased regularly as the present one-dimensional case. For example, in Fig. 4, we see that for ϵ_3 , there are two nodes if $x \in (-\infty, \infty)$ and one node if x (or r) $\in [0, \infty)$.

Appendix A. Numerical Method to Solve Schrödinger Equation (3)

In this Appendix, we are going to show how to solve Schrödinger equation (3) numerically via software Mathematica. For given values of the parameters a and b as well as the integral interval, say x_r , we just use the Mathematica commands in the following six steps to obtain the eigenvalues and the corresponding normalized eigenfunctions

- (1) `pfun=ParametricNDSolveValue[{-1/2 y''[x] + v[x]y[x] == λy[x], y[-xr] == 0, y'[-xr] == 1}, y, {x, -xr, xr}, {λ}]`,
- (2) `Plot[Evaluate@Table[pfun[λ][x], {λ, -10, 25, 1}], {x, -xr, xr}, PlotRange → {-1, 1}]`,
- (3) `Plot[pfun[λ][xr], {λ, -10, 25}, PlotRange → {-10, 10}]`,
- (4) `Map[FindRoot[pfun[λ][xr], {λ, #}], &, {0.5, 2.3, 4.5, ...}]`,
- (5) `Ei=λ /. %`,
- (6) `Plot[Evaluate@Table[$\frac{\text{pfun}[\lambda][x]}{\sqrt{\text{NIntegrate}[(\text{pfun}[\lambda][x])^2, \{x, -x_r, x_r\}]^{1/2}}}$, {λ, Ei}], {x, -xr, xr}, PlotRange → {-1, 1}]`.

I hope this Appendix will be helpful to the readers who are familiar with the Mathematica.

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