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# General properties of pseudo-contractibility

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General facts about pseudo-homotopies and pseudo-contractibility are studied for topological spaces and continua. As a consequence of these, we find several conditions that obstruct pseudo-contractibility and we present examples of pseudo-contractible continua and non-pseudo-contractible continua.



## 1. Introduction

The concept of pseudo-contractibility was introduced by R. H. Bing. However, W. Kuperberg gave the first example which proves that the notions of pseudo-contractibility and contractibility are different. This example was never published by himself but it is known among continuum theorists. He also asked whether or not the space $\sin\left(\frac{1}{x}\right)$ curve is pseudo-contractible (see [12]). H. Katsuura proves in [8] that the space $\sin\left(\frac{1}{x}\right)$ curve is not pseudo-contractible with factor space itself. In the same paper he proves that if the factor space $Y$ is a nondegenerate indecomposable continuum such that each one of their composants is arc-wise connected, and if $X$ is a continuum having a proper nondegenerate arc component, then $X$ is

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not pseudo-contractible with factor space $Y$. After that, W. Dębski proves in [5] that the space $\sin(\frac{1}{x})$ curve is not pseudo-contractible. On the other hand, M. Sobolewski in [17] shows that the only (up homeomorphism) pseudo-contractible chainable continuum is the arc. This shows that the pseudo-arc is not pseudo-contractible, answering Problem 118 of [12]. The interested reader is referred to [1], [7], [8], [12] and [17] for getting more information about these results.

This paper is divided in nine sections. After preliminaries, we give, in sections three and four, several and general facts about pseudo-homotopies and pseudo-contractibility. In section five, pseudo-contractibility with respect to a topological space is studied. The concept pseudo-homotopy equivalent is related with pseudo-contractibility in section six. In sections seven and eight we give conditions which imply nonpseudo-contractibility. Finally in section nine we present some open questions about it.

## 2. Preliminaries

A *continuum* means a nonempty compact connected metric space. A topological space is said to be *continuumwise connected* provided that any two of its points are contained in a proper subcontinuum of the space. A *mapping* means a continuous function. Let $X$ and $Y$ topological spaces, we write $X \approx Y$ if $X$ is homeomorphic to $Y$. An *arc* is understood as a homeomorphic image of the closed unit interval $I = [0, 1]$. If any two points of a space can be joined by an arc lying in the space, then the space is said to be *arcwise-connected*.

Let $X$ and $Y$ be topological spaces. The symbol $C(X, Y)$ denotes the topological space of all mappings from $X$ to $Y$ endowed with the compact-open topology. It is well known that if $X$ is compact and $Y$ is a compact metric space, then the compact-open topology coincides with the topology given by the supremum metric on $C(X, Y)$.

Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be topological spaces such that $X_1 \cap X_2 = \emptyset$. The *free union* of $X_1$ and $X_2$ is the topological space $(X, \tau)$, where $X = X_1 \cup X_2$ and $U \in \tau$ if and only if $U \cap X_i \in \tau_i$ for each $i = 1, 2$. The free union of $X_1$ and $X_2$ is denoted by $X_1 + X_2$. If $A$ is a non-empty closed subset of $X_1$, $f : A \to X_2$ is a mapping and $D$ is the partition of $X_1 + X_2$ given by $D = \{\{p\} \cup f^{-1}(p) : p \in f(A)\} \cup \{\{x\} : x \in X_1 + X_2 \setminus (A \cup f(A))\}$, the decomposition space thus obtained is denoted by $X_1 \cup_f X_2$ and it is called the *attached space*. If $X$ and $Y$ are disjoint continua, then the attached space $X \cup_f Y$ is a continuum ([15, Theorem 3.20]).

## 3. Pseudo-homotopy

In this section we will develop general facts concerning pseudo-homotopies.

**Definition 1.** Let $X$ and $Y$ be topological spaces and let $f, g : X \to Y$ be mappings. We say that $f$ is *homotopic* to $g$ (or $f$ and $g$ are homotopic, written by $f \simeq g$), if there exists a mapping $H : X \times I \to Y$ (where $I$ is the unit interval), called *homotopy*, fulfilling $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each $x \in X$.

**Definition 2.** Let $X$ and $Y$ be topological spaces and let $f, g : X \to Y$ be mappings. We say that $f$ is *pseudo-homotopic* to $g$ (or $f$ and $g$ are pseudo-homotopic) if there exist a continuum $C$, points $a, b \in C$ and a mapping $H : X \times C \to Y$ fulfilling $H(x, a) = f(x)$ and $H(x, b) = g(x)$ for each $x \in X$. The continuum $C$ is called *factor space*. The mapping $H$ is called a *pseudo-homotopy* between $f$ and $g$. We write $f \simeq_C g$ to say that $f$ is pseudo-homotopic to $g$, where $C$ denotes a factor space.

It is easy to verify that if $f \simeq_C g$ and there exist a continuum $K$, and an onto mapping from $K$ to $C$, then $f \simeq_K g$. Moreover, if there are a continuum $K'$ and an onto mapping from $K'$ to some subcontinuum $C' \subset C$ such that $a, b \in C'$, then $f \simeq_{K'} g$. Recall that two continua $X$ and $Y$ are said to be *continuously equivalent* provided that there are two onto mappings $f : X \to Y$ and $g : Y \to X$. So if $C$ and $D$ are continuously equivalent continua. Then $f \simeq_C g$ if and only if $f \simeq_D g$. In particular if $C_1 \approx C_2$, then $f \simeq_{C_1} g$ if and only if $f \simeq_{C_2} g$. On the other hand if $f \simeq_C g$ and $K$ is a subcontinuum of $C$ such that $a, b \in K$, then $f \simeq_K g$. In particular, if $I_{ab}$ is an irreducible continuum between $a$ and $b$ contained in $C$, then $f \simeq_{I_{ab}} g$. Moreover if $I_{ab}$ is an arc from $a$ to $b$, then $f \simeq g$. In particular $f \simeq_C g$ implies $f \simeq g$ if $C$ is arcwise-connected. Finally, it is easy to see that if $f,\ g : X \to Y$ are mappings such that $f$ is pseudo-homotopic to $g$ and $Z$ is a subset of $X$, then $f|_Z$ is pseudo-homotopic to $g|_Z$.

We will give an equivalence relation in $C(X, Y)$ as follows. Let $f$, $g$ in $C(X, Y)$. We say that $f$ is related to $g$ if and only if there is a continuum $K$, such that $f \simeq_K g$. We write $f \simeq_* g$ in order to say that $f$ is related to $g$.

**Theorem 3.** *The relation $\simeq_*$ is an equivalence relation in $C(X, Y)$.*

**Proof.** The reflexive and symmetric properties are immediate.

Let us just to prove transitivity. Let $f,\ g,\ h : X \to Y$ be mappings, such that $f \simeq_* g$ and $g \simeq_* h$. Then there exist continua $C_1$, $C_2$, points $a_1,\ b_1 \in C_1$, points $a_2,\ b_2 \in C_2$ and mappings $H_1 : X \times C_1 \to Y$ and $H_2 : X \times C_2 \to Y$ fulfilling $H_1(x, a_1) = f(x)$, $H_1(x, b_1) = g(x)$ and $H_2(x, a_2) = g(x)$, $H_2(x, b_2) = h(x)$ for each $x \in X$ respectively. Without loss of generality, we assume that $C_1 \cap C_2 = \emptyset$. We consider $j : \{b_1\} \to C_2$ given by $j(b_1) = a_2$ and $D = C_1 \cup_j C_2$. We define a function $H : X \times D \to Y$ by

$$H(x, d) = \begin{cases} H_1(x, d) & \text{if} \quad d \in C_1 \\ H_2(x, d) & \text{if} \quad d \in C_2. \end{cases}$$

It is clear that $H$ is a pseudo-homotopy between $f$ and $h$. □

The equivalence classes in $C(X, Y)$ under the relation $\simeq_*$ are called pseudo-homotopy classes.

**Theorem 4.** *Let $X$ and $Y$ be compact metric spaces and let $f,\ g : X \to Y$ be mappings. The mappings $f$ and $g$ are pseudo-homotopic if and only if there exist a continuum in $C(X, Y)$ containing $f$ and $g$.*

**Proof.** Suppose $f \simeq_C g$. For every $c \in C$, we define the mapping $h_c : X \to Y$ given by $h_c(x) = H(x, c)$, where $H$ is the pseudo-homotopy between $f$ and $g$. Then the function $G : C \to C(X, Y)$ defined by $G(c) = h_c$ is continuous. Since $C(X, Y)$ is a Hausdorff space and $G(C) \subset C(X, Y)$, the image $G(C)$ is a Hausdorff space. Hence $G(C)$ is metrizable ([11, §41, VI, Theorem 3]). So, $G(C)$ is a continuum containing $f$ and $g$.

Conversely, let $f,\ g \in C(X, Y)$ and let $H \subset C(X, Y)$ be a continuum containing $f$ and $g$. The function $F : X \times H \to Y$ given by $F(x, h) = h(x)$ is continuous and it satisfies $F(x, f) = f(x)$ and $F(x, g) = g(x)$ for all $x \in X$. □

In this sense every pseudo-homotopy class is continuumwise connected.

**Corollary 5.** *Let $X$, $Y$ be compact metric spaces. Every pair of mappings $f,\ g : X \to Y$ are pseudo-homotopic if and only if the space $C(X, Y)$ is continuumwise connected.*

Regarding the composition of functions, we have the following results.

**Theorem 6.** *Let $h : Y \to Z$, $k : W \to X$ and $f,\ g : X \to Y$ be mappings. If $f \simeq_C g$, then $h \circ f \simeq_C h \circ g$ and $f \circ k \simeq_C g \circ k$.*

**Proof.** Since $f \simeq_C g$, there exist points $a,\ b \in C$ and a mapping $H : X \times C \to Y$ such that $H(x, a) = f(x)$ and $H(x, b) = g(x)$ for each $x \in X$.

To prove the first part we consider the function $G : X \times C \to Z$ defined by $G(x, c) = (h \circ H)(x, c)$. The function $G$ is a pseudo-homotopy between $h \circ f$ and $h \circ g$.

On the other hand, the function $F : W \times C \to Y$ given by $H(z, c) = F(k(z), c)$ is a pseudo-homotopy between $f \circ k$ and $g \circ k$. □

**Theorem 7.** *Let $f,\ f' : X \to Y$ and $g,\ g' : Y \to Z$ be mappings such that $f \simeq_{C_1} f'$ and $g \simeq_{C_2} g'$. Then the composition $g \circ f$ is pseudo-homotopic to the composition $g' \circ f'$.*

**Proof.** By hypothesis, there are points $a_1,\ b_1 \in C_1,\ a_2,\ b_2 \in C_2$ and mappings $H_1 : X \times C_1 \to Y$ and $H_2 : Y \times C_2 \to Z$ such that $H_1(x, a_1) = f(x)$, $H_1(x, b_1) = f'(x)$ for each $x \in X$ and $H_2(y, a_2) = g(y)$, $H_2(y, b_2) = g'(y)$ for each $y \in Y$. Consider the continuum $C = C_1 \times C_2$ and the points $\hat{a_0} = (a_1, a_2)$, $\hat{b_0} = (b_1, b_2) \in C_1 \times C_2$, then the function $F : X \times (C_1 \times C_2) \to Z$ defined by $F(x, (c_1, c_2)) = H_2(H_1(x, c_1), c_2)$ is a pseudo-homotopy between $g \circ f$ and $g' \circ f'$. □

Let $\{X_j\}_{j\in J}$ be a family of topological spaces and let $\prod_{j\in J} X_j$ the product space endowed with the product topology. Recall that the for each natural number $i$, the map $\pi_i : \prod_{j\in J} X_j \to X_i$ is defined by $\pi_i((x_j)_{j\in J}) = x_i$.

The following result follows from Theorem 6.

**Theorem 8.** *Let $\{Y_\alpha\}_{\alpha\in I}$ be a family of topological spaces. Let $f,\ g : X \to \prod_{\alpha\in I} Y_\alpha$ be mappings. If $f$ is pseudo-homotopic to $g$, then so are $\pi_\alpha \circ f$ and $\pi_\alpha \circ g$.*

**Corollary 9.** *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a family of topological spaces. Let $f,\ g : X \to \prod_{n\in\mathbb{N}} Y_n$ be mappings. The mappings $f$ and $g$ are pseudo-homotopic if and only if the mappings $\pi_n \circ f$ and $\pi_n \circ g$ are pseudo-homotopic.*

**Proof.** Let $f,\ g : X \to \prod_{n\in\mathbb{N}} Y_n$ be mappings. Suppose that $\pi_n \circ f$ and $\pi_n \circ g$ are pseudo-homotopic for each $n \in \mathbb{N}$. We have for every $n \in \mathbb{N}$, there exist a continuum $C_n$, points $a_n,\ b_n \in C_n$ and a mapping $H_n : X \times C_n \to Y_n$ such that $H_n(x, a_n) = (\pi_n \circ f)(x)$ and $H_n(x, b_n) = (\pi_n \circ g)(x)$. Notice that $C = \prod_{n\in\mathbb{N}} C_n$ is a continuum. Let $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}} \in C$. The function $H : X \times C \to \prod_{n\in\mathbb{N}} Y_n$ defined by $H(x, (c_n)_{n\in\mathbb{N}}) = (H_n(x, c_n))_{n\in\mathbb{N}}$ is a pseudo-homotopy between $f$ and $g$.

The converse is immediate from Theorem 8. □

## 4. Pseudo-contractibility

In this part we will give general facts about pseudo-contractibility. Let us start with the usual definition of contractibility.

**Definition 10.** A topological space $X$ is said to be *contractible* if its identity mapping is homotopic to a constant mapping in $X$, i.e., there exists a mapping $H : X \times [0, 1] \to X$ satisfying $H(x, 0) = x$ and $H(x, 1) = x_0$, for each $x \in X$.

**Definition 11.** A topological space $X$ is said to be *pseudo-contractible* if its identity mapping is pseudo-homotopic to a constant mapping into $X$, i.e., there exist a continuum $C$, points $a,\ b \in C$, $x_0 \in X$ and a mapping $H : X \times C \to X$ fulfilling $H(x, a) = x$ and $H(x, b) = x_0$ for each $x \in X$.

Notice that $X$ is (pseudo-)contractible if and only if each mapping $f : X \to X$ is (pseudo-)homotopic to a constant mapping.

The following example was given by W. Kuperberg and it was the first example showing that the concepts of contractibility and pseudo-contractibility are different. We describe and draw here this example for the interested readers (see Fig. 1).

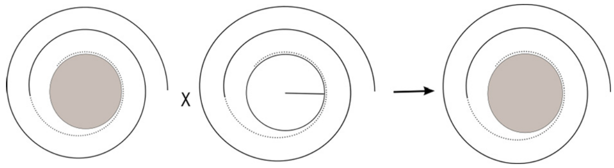


**Fig. 1.** Pseudo-contractible continuum.

**Example 12** *(W. Kuperberg).* Let $\mathbb{C}$ be the complex plane and let $X_0 = \{\frac{t+2}{t+1}e^{it} : t \in [0, \infty)\}$ be the spiral approaching the unit circle $S^1$. Let $X = X_0 \cup \{x : |x| \leq 1\} \subset \mathbb{C}$. We observe that the continuum $X$ is not contractible because it is not arc-wise connected.

Consider $C = X_0 \cup \{x : |x| = 1\} \cup X_1 \subset \mathbb{C}$, where $X_1 = \{x \in \mathbb{C} : Im(x) = 0,\ 0 \leq Re(x) \leq 1\}$.

We define a mapping $H : X \times C \to X$ as follows:

1. $H(\frac{t+2}{t+1}e^{it}, \frac{t'+2}{t'+1}e^{it'}) = \frac{t+t'+2}{t+t'+1}e^{i(t+t')}$ if $t,\ t' \in [0, \infty)$.
2. $H(x, \frac{t+2}{t+1}e^{it}) = xe^{it}$ if $|x| \leq 1,\ t \in [0, \infty)$.
3. $H(x, x') = xx'$ if $|x| \leq 1,\ |x'| = 1$ or $x' \in X_1$.
4. $H(\frac{t+2}{t+1}e^{it}, x) = xe^{it}$ if $t \in [0, \infty),\ |x| = 1$ or $x \in X_1$.

We have that $H(x, 2) = x$ and $H(x, 0) = 0$ for each $x \in X$. So, $X$ is pseudo-contractible.

As a consequence of the comments after of Definition 2 and Urysohn's Lemma, we have the following four results.

**Theorem 13.** *If a continuum $X$ is pseudo-contractible with (locally connected continuum) arcwise-connected continuum as factor space, then $X$ is contractible.*

**Corollary 14.** *If $X$ is a pseudo-contractible continuum with factor space $C$ and $f : C' \to C$ is an onto mapping, then $X$ is pseudo-contractible with factor space $C'$.*

**Corollary 15.** *Let $C_1$ and $C_2$ be continua such that $C_1$ is continuously equivalent to $C_2$. Hence $X$ is pseudo-contractible with factor space $C_1$ if and only if $X$ is pseudo-contractible with factor space $C_2$.*

**Corollary 16.** *If a topological space $X$ is contractible, then $X$ is pseudo-contractible with any continuum as factor space.*

From Hahn–Mazurkiewicz's Theorem, every locally connected continuum is the continuous image to the interval $[0, 1]$. By Urysohn's Lemma, there exist mappings from every normal space to the interval $[0, 1]$. So, each locally connected continuum $C$ is continuously equivalent to the interval.

In this way, we have the following.

**Theorem 17.** *Let $X$ be a topological space, the following are equivalent:*

1. *$X$ is pseudo-contractible with any continuum as factor space.*
2. *$X$ is pseudo-contractible with any locally connected continuum $C$ as factor space.*
3. *$X$ is pseudo-contractible with some locally connected continuum $C$ as factor space.*
4. *$X$ is pseudo-contractible with some arcwise-connected continuum as factor space.*
5. *$X$ is pseudo-contractible with any arcwise-connected continuum as factor space.*
6. *$X$ is pseudo-contractible with some factor space $C$ such that $a$ and $b$ can be joined with an arc in $C$, where $C$, $a$ and $b$ are as in Definition 11.*
7. *$X$ is contractible.*

**Definition 18.** Let $X$ be a topological space and let $A$ be a closed subset of $X$. A *retraction* from $X$ onto $A$ is a mapping $r : X \to A$ such that $r(a) = a$ for each $a \in A$. The set $A$ is called a *retract* of $X$.

We will see that pseudo-contractibility (as well as contractibility) is preserved under retractions.

**Theorem 19.** *Let $X$ be a pseudo-contractible space. If $A$ is a retract of $X$, then $A$ is pseudo-contractible.*

**Proof.** Let $C$ a continuum, let $a,\ b \in C$, $x_0 \in X$ and let $H : X \times C \to X$ a mapping satisfying $H(x, a) = x$ and $H(x, b) = x_0$ for each $x \in X$. Since $A$ is a retract of $X$, there exists a mapping $r : X \to A$ such that $r(y) = y$ for each $y \in A$. Let $a_0 = r(x_0) \in A$. Consider the mapping $i : A \times C \to X \times C$ given by $i(y, c) = (y, c)$.

In order to show that $A$ is pseudo-contractible consider the mapping $G : A \times C \to A$ defined by $G(y, c) = (r \circ H \circ i)(y, c)$. The function $G$ is a pseudo-homotopy between the identity mapping and a constant mapping. □

**Remark 20.** Notice that pseudo-contractibility is a topological property.

**Theorem 21.** *Let $X$ and $Y$ be topological spaces. The spaces $X$ and $Y$ are pseudo-contractible if and only if the product space $X \times Y$ is pseudo-contractible.*

**Proof.** If $X$ and $Y$ are pseudo-contractible, there exist continua $C_1,\ C_2$, points $a_1,\ b_1 \in C_1,\ a_2,\ b_2 \in C_2$, $x_0 \in X$, $y_0 \in Y$ and mappings $H_1 : X \times C_1 \to X$, $H_2 : Y \times C_2 \to Y$ fulfilling $H_1(x, a_1) = x$, $H_1(x, b_1) = x_0$ and $H_2(y, a_2) = y$, $H_2(y, b_2) = y_0$ for each $x \in X$ and each $y \in Y$. Consider the continuum $C_1 \times C_2$ and the points $(a_1, a_2),\ (b_1, b_2) \in C_1 \times C_2$. The function $H : (X \times Y) \times (C_1 \times C_2) \to X \times Y$ defined by $H((x, y), (c_1, c_2)) = (H_1(x, c_1), H_2(y, c_2))$ is a pseudo-homotopy between the identity mapping and the constant mapping whose image is $(x_0, y_0)$.

Now suppose that $X \times Y$ is pseudo-contractible, let $(x_0, y_0) \in X \times Y$ of image of the constant mapping satisfying the definition of pseudo-contractibility. Since $X \times \{y_0\} \approx X$ and $\{x_0\} \times Y \approx Y$, and $X \times \{y_0\}$ and $\{x_0\} \times Y$ are retracts of $X \times Y$, Theorem 19 and Remark 20 imply that $X$ and $Y$ are pseudo-contractible. □

**Corollary 22.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of topological spaces. The space $X_n$ is pseudo-contractible for all $n \in \mathbb{N}$ if and only if the product space $\prod_{n\in\mathbb{N}} X_n$ is pseudo-contractible.*

**Proof.** Suppose $X_n$ is pseudo-contractible for all $n \in \mathbb{N}$, hence there exist $\{C_n\}_{n\in\mathbb{N}}$ a sequence of continua, points $a_n,\ b_n \in C_n$, $x_n^0 \in X_n$ and mappings $H_n : X_n \times C_n \to X_n$, satisfying $H_n(x, a_n) = x$, $H_n(x, b_n) = x_n^0$ for each $x \in X_n$ and each $n \in \mathbb{N}$. Consider the continuum $C = \prod_{n\in\mathbb{N}} C_n$ and the points $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in C$. We define the function $H : (\prod_{n\in\mathbb{N}} X_n) \times C \to \prod_{n\in\mathbb{N}} X_n$ by $H((x_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}) = (H_n(x_n, c_n))_{n\in\mathbb{N}}$. The mapping $H$ is a pseudo-homotopy between the identity mapping and the constant mapping whose image is $(x_n^0)_{n\in\mathbb{N}}$.

Now assume that $\prod_{n\in\mathbb{N}} X_n$ is pseudo-contractible, let $(x_n^0)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ of image of the constant mapping satisfying the definition of pseudo-contractibility. Note that $X_n \times \prod_{j\neq n}\{x_j^0\} \approx X_n$ and $X_n \times \prod_{j\neq n}\{x_j^0\}$ is a retract of $\prod_{n\in\mathbb{N}} X_n$, for each $n \in \mathbb{N}$. It follows from Theorem 19 and Remark 20 that $X_n$ is pseudo-contractible for each $n \in \mathbb{N}$. □

As a consequence of Theorem 19 and Remark 20 we have the following results.

**Corollary 23.** *Let $\{X_\alpha\}_{\alpha\in I}$ be a family of topological spaces. If $\prod_{\alpha\in I} X_\alpha$ is pseudo-contractible, then $X_\alpha$ is pseudo-contractible for each $\alpha \in I$.*

**Corollary 24.** *Let $X$ be a topological space. The following five statements are equivalent:*

1. *$X$ is pseudo-contractible.*
2. *$X^n$ is pseudo-contractible for each $n \in N$.*
3. *$X^n$ is pseudo-contractible for some $n \in N$.*
4. *The cylinder $X \times [0,1]$ is pseudo-contractible.*
5. *$\prod_{n\in\mathbb{N}} X_n$ is pseudo-contractible, where $X_n = X$ for each $n \in \mathbb{N}$.*

## 5. Pseudo-contractibility with respect to $Y$

**Definition 25.** Let $X$ and $Y$ be topological spaces. We say that $X$ is *(pseudo-)contractible with respect to $Y$* if each mapping $f : X \to Y$ is (pseudo-)homotopic to a constant mapping.

**Definition 26.** A subspace $Z$ of $X$ is said to be *(pseudo-)contractible in $X$* if the inclusion mapping into $X$, is (pseudo-)homotopic to a constant mapping in $X$.

Note that if $Z \subset X$ and $Z$ is (pseudo-)contractible with respect to $X$, then $Z$ is (pseudo-)contractible in $X$.

**Theorem 27.** *If $X$ is pseudo-contractible with respect to $Y$ and $Y$ is continuumwise connected, then every pair of mappings from $X$ into $Y$ are pseudo-homotopic. In particular this holds if $X$ is pseudo-contractible with respect to $Y$ and $Y$ is a continuum.*

**Proof.** Let $f,\ g : X \to Y$ be mappings. Since $X$ is pseudo-contractible with respect to $Y$, there exist mappings $H_1 : X \times C_1 \to Y$ and $H_2 : X \times C_2 \to Y$, points $a_1,\ b_1 \in C_1$ and $a_2,\ b_2 \in C_2$ such that $H_1(x, a_1) = f(x)$, $H_1(x, b_1) = y_1$ and $H_2(x, a_2) = y_2$, $H_2(x, b_2) = g(x)$. Since $Y$ is continuumwise connected, there exists a continuum $K$ joining $y_1$ and $y_2$. Now we consider the attached continuum $C = C_1 \cup_j K \cup_l C_2$, where $j : \{b_1\} \to K$, $l : \{y_2\} \to C_2$ are mappings defined by $j(b_2) = y_1$ and $l(y_2) = a_2$. Define the mapping $F : X \times C \to Y$ as follows,

$$F(x,c) = \begin{cases} H_1(x,c) & \text{if} \quad c \in C_1 - \{b_1\}, \\ y_1 & \text{if} \quad c = \{b_1, y_1\}, \\ c & \text{if} \quad c \in K - \{y_1, y_2\}, \\ y_2 & \text{if} \quad c = \{a_2, y_2\}, \\ H_2(x,c) & \text{if} \quad c \in C_2 - \{a_2\}. \end{cases}$$

It can be checked that $F$ is a pseudo-homotopy between $f$ and $g$. □

Note that if $X$ is a topological space, $Y$ is continuumwise connected and $X$ is pseudo-contractible with respect to $Y$, then every pair of constant mappings from $X$ into $Y$ are pseudo-homotopic with factor space $Y' \subset Y$, where $Y'$ is a subcontinuum containing the image of both constant mappings. In this case the projection mapping of the product $X \times Y'$ to $Y'$ is the pseudo-homotopy. On the other hand, if $Z$ is a continuumwise connected pseudo-contractible space, then its identity mapping is pseudo-homotopic to any constant mapping. In particular these results hold when $Y$ and $Z$ are continua.

**Corollary 28.** *Let $X$ be a compact metric space and let $Y$ be a (continuum) continuumwise connected space. $X$ is pseudo-contractible with respect to $Y$ if and only if the space $C(X,Y)$ is continuumwise connected.*

**Proof.** Let $f,\ g : X \to Y$ be mappings. Since $X$ is pseudo-contractible with respect to $Y$, Theorem 27 implies $f \simeq_C g$. From Theorem 4, there exists a continuum in $C(X,Y)$ joining $f$ with $g$.

Conversely, we take two mappings $f,\ g : X \to Y$ where $g$ is a constant mapping. By hypothesis there exists a continuum $K$ in $C(X,Y)$ joining $f$ with $g$. We apply Theorem 4 to get that $X$ is pseudo-contractible with respect to $Y$. □

**Theorem 29.** *Let $X$ be a continuum (continuumwise connected space) the following statements are equivalent.*

1. *$X$ is pseudo-contractible.*
2. *For each compact metric space $Y$, $X$ is pseudo-contractible with respect to $Y$.*
3. *For each compact metric space $Z$, $Z$ is pseudo-contractible with respect to $X$.*
4. *$C(X,X)$ is continuumwise connected.*

**Proof.** (1) ⇒ (2). Let $Y$ be a compact metric space and let $f : X \to Y$ be a mapping. Since $X$ is pseudo-contractible, the identity mapping is pseudo-homotopic to a constant mapping. By Theorem 6, we have that $f = f \circ id_X$ is pseudo-homotopic to a constant mapping. Hence $X$ is pseudo-contractible with respect to $Y$.

(1) ⇒ (3). Let $Z$ be a compact metric space and let $g : Z \to X$ be a mapping. Since $X$ is pseudo-contractible, the identity mapping is pseudo-homotopic to a constant mapping. By Theorem 6, $g = id_X \circ g$ is pseudo-homotopic to a constant mapping, so $Z$ is pseudo-contractible with respect to $X$.

The implications (2) ⇒ (1) and (3) ⇒ (1) are immediate.

Since $X$ is a continuum, by Corollary 28 we have (1) if and only if (4). □

**Remark 30.** Notice that if $X$ is pseudo-contractible, then the space $C(X,X)$ is continuumwise connected and therefore connected. We do not if the condition $C(X,X)$ connected implies $X$ is pseudo-contractible.

**Theorem 31.** *Let $X$ and $Y$ be topological spaces. If $X$ is pseudo-contractible with respect to $Y$ and $A$ is a retract of $X$, then $A$ is pseudo-contractible with respect to $Y$.*

**Proof.** Let $f : A \to Y$ be a mapping and let $r : X \to A$ be a retraction from $X$ to $A$. Since $X$ is pseudo-contractible with respect to $Y$, then $f \circ r : X \to Y$ is pseudo-homotopic to a constant mapping. On the other hand, we know that the restriction of two mappings are homotopic if the mappings are homotopic, so $f = (f \circ r)|_A$ is pseudo-homotopic to a constant mapping. □

In the following sections we will obtain several obstructions to pseudo-contractibility. So, it is possible to get new pseudo-contractible continua and new non pseudo-contractible continua.

## 6. Pseudo-homotopically equivalent and pseudo-contractibility

**Definition 32.** Let $X$ and $Y$ be topological spaces. We say that $Y$ is *semi-homotopy equivalent to* $X$, written $Y \approx^{SE} X$, if there exist two mappings $g : Y \to X$ and $f : X \to Y$ such that $f \circ g \simeq id_Y$.

**Definition 33.** Let $X$ and $Y$ be topological spaces. We say that $Y$ *is semi-pseudo-homotopy equivalent to* $X$, written $Y \approx_P^{SE} X$, if there exist a continuum $C$ and two mappings $g : Y \to X$ and $f : X \to Y$ such that $f \circ g \simeq_C id_Y$.

When the factor space $C$ is homeomorphic to the interval $I = [0,1]$, $Y \approx_P^{SE} X$ is equal to $Y \approx^{SE} X$.

**Theorem 34.** *Let $Y$ and $X$ be topological spaces. If $X$ is pseudo-contractible and $Y$ is semi-pseudo-homotopy equivalent to $X$, then so is $Y$.*

**Proof.** By hypothesis, we have that $id_X \simeq_C x_0$, where $x_0$ is a constant mapping, also there exist two mappings $g : Y \to X$ and $f : X \to Y$ and a continuum $K$, such that $f \circ g \simeq_K id_Y$. By Theorem 6, $f = f \circ id_X \simeq_C f \circ x_0$. Notice that $y_0 = f \circ x_0 : X \to Y$ is a constant mapping.

Therefore, by Theorem 6, we have that $f \circ g \simeq_C y_0 \circ g$, where $y_0 \circ g : Y \to Y$ is a constant mapping. Since $id_Y \simeq_K f \circ g$ and $f \circ g \simeq_C y_0 \circ g$, by Theorem 3, there exists a continuum $D$ satisfying $id_Y \simeq_D y_0 \circ g$. Hence $Y$ is pseudo-contractible. □

**Theorem 35.** *Let $X$, $Z$ and $Y$ be topological spaces. If $X$ is pseudo-contractible with respect to $Y$ and $Z \approx_P^{SE} X$, then $Z$ is pseudo-contractible with respect to $Y$.*

**Proof.** Let $h : Z \to Y$ be a mapping. Since $Z \approx_P^{SE} X$ there are mappings $f : Z \to X$ and $g : X \to Z$ such that $g \circ f \simeq_K id_Z$. Then, by Theorem 6, $h \circ g \circ f \simeq_K h \circ id_Z = h$.

On the other hand, since $X$ is pseudo-contractible with respect to $Y$, $h \circ g \simeq_C y_0$, where $y_0$ is a constant mapping. Hence, by Theorem 6, $h \circ g \circ f \simeq_C y_0 \circ f = y_0$. Therefore, by Theorem 3, $h \simeq_D y_0$. □

**Theorem 36.** *Let $X$, $Z$ and $Y$ be topological spaces. If $X$ is pseudo-contractible with respect to $Y$ and $Z \approx_P^{SE} Y$, then $X$ is pseudo-contractible with respect to $Z$.*

**Proof.** Let $h : X \to Z$ be a mapping. Since $Z \approx_P^{SE} Y$ there exist mappings $f : Z \to Y$ and $g : Y \to Z$ such that $g \circ f \simeq_K id_Z$. Then by Theorem 6, $g \circ f \circ h \simeq_K id_Z \circ h = h$.

On the other hand, since $X$ is pseudo-contractible with respect to $Y$, $f \circ h \simeq_C y_0$. Hence by Theorem 6, $g \circ f \circ h \simeq_C g \circ y_0 = g(y_0)$. Therefore, Theorem 3 implies that $h \simeq_D g(y_0)$. □

**Definition 37.** Let $X$ and $Y$ be topological spaces. It is said that $X$ and $Y$ are *homotopically equivalent* (or have the same homotopy type), written $X \approx^E Y$, if there exist two mappings $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

**Definition 38.** Let $X$ and $Y$ be topological spaces. It is said that $X$ and $Y$ are *pseudo-homotopically* equivalent (or have the same pseudo-homotopy type), written $X \approx_P^E Y$, if there exist two mappings $f : X \to Y$ and $g : Y \to X$ and two continua $K$ and $C$ such that $f \circ g \simeq_K id_Y$ and $g \circ f \simeq_C id_X$.

When the factor spaces $C$ and $K$ are homeomorphics to the interval $I = [0, 1]$, $Y \approx_P^E X$ is equals $Y \approx^E X$. As a corollary of Theorem 34 we get the following.

**Corollary 39.** *Let $X$ and $Y$ be topological spaces. If $X \approx_P^E Y$ and one of them is pseudo-contractible, then the other one is pseudo-contractible.*

The following theorem is easy to prove.

**Theorem 40.** *Let $X$ be a topological space. $X$ is pseudo-contractible if and only if $X$ has the same pseudo-homotopy type as a point $p$.*

Note that Definition 33 gives us a relation within the family of all topological spaces. We say that $X \sim Y$ if and only if $Y \approx_P^E X$.

**Proposition 41.** *The relation $\sim$ is an equivalence relation.*

**Proof.** The reflexive and symmetric properties are immediate.

If $Y \approx_P^E X$, then there exist mappings $g : Y \to X$ and $f : X \to Y$ and continua $C$ and $D$, such that $f \circ g \simeq_C id_Y$ and $g \circ f \simeq_D id_X$. In the same way if $Z \approx_P^E Y$, there exist mappings $h : Z \to Y$ and $j : Y \to Z$

and continua $E$ and $F$ such that $j \circ h \simeq_E id_Z$ and $h \circ j \simeq_F id_Y$. Let $H : Z \to X$ the mapping given by $H(z) = (g \circ h)(z)$ and $G : X \to Z$ is a mapping defined by $G(x) = (j \circ f)(x)$. By hypothesis we have that $h \circ j \simeq_F id_Y$. Applying Theorem 6, we obtain $(h \circ j) \circ f \simeq_F f$ and $g \circ (h \circ j) \circ f \simeq_F g \circ f$. Since $g \circ f \simeq_D id_X$, Theorem 3 guarantees a continuum $M$ such that $H \circ G = g \circ h \circ j \circ f \simeq_M id_X$. In the same way, it can be proved $G \circ H \simeq_N id_Z$. □

Concerning to the product of topological spaces we have the following result.

**Proposition 42.** *Let $\{X_n\}_{n\in\mathbb{N}}$, $\{Y_n\}_{n\in\mathbb{N}}$ be families of compact metric spaces. If $X_n \approx_P^E Y_n$ for each $n \in \mathbb{N}$ then $\prod_{n\in\mathbb{N}} X_n \approx_P^E \prod_{n\in\mathbb{N}} Y_n$.*

**Proof.** For each $n \in \mathbb{N}$, there exist continua $C_n$, $D_n$ and maps $f_n : X_n \to Y_n$ and $g_n : Y_n \to X_n$ such that $f_n \circ g_n \simeq_{C_n} id_{Y_n}$ and $g_n \circ f_n \simeq_{D_n} id_{X_n}$. That means, for each $n \in \mathbb{N}$ we have mappings $H_n : X_n \times C_n \to X_n$ and $G_n : Y_n \times D_n \to Y_n$ such that $H_n(x, a_n) = (f_n \circ g_n)(x)$ and $H_n(x, b_n) = x$ and $G_n(y, a'_n) = (g_n \circ f_n)(y)$ and $G_n(y, b'_n) = y$ respectively. Let $f : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} Y_n$ and $g : \prod_{n\in\mathbb{N}} Y_n \to \prod_{n\in\mathbb{N}} X_n$ given by $f((x_n)_{n\in\mathbb{N}}) = (f_n(x_n))_{n\in\mathbb{N}}$ and $g((y_n)_{n\in\mathbb{N}}) = (g_n(y_n))_{n\in\mathbb{N}}$ respectively. Let $C = \prod_{n\in\mathbb{N}} C_n$, $\hat{a} = (a_n)_{n\in\mathbb{N}}, \hat{b} = (b_n)_{n\in\mathbb{N}}$ we will prove that $f \circ g \simeq_C id_Y$. We define the mapping $\hat{H} : \prod_{n\in\mathbb{N}} X_n \times C \to \prod_{n\in\mathbb{N}} X_n$ by $\hat{H}((x_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}) = (H_n(x_n, c_n))_{n\in\mathbb{N}}$.

We have that $\hat{H}((x_n)_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}}) = (H_n(x_n, a_n))_{n\in\mathbb{N}} = ((f_n \circ g_n)(x_n))_{n\in\mathbb{N}} = (f \circ g)(\hat{x})$ and $\hat{H}((x_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = (H_n(x_n, b_n))_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}} = \hat{x}$.

Analogously it can be proved that $g \circ f \simeq_D id_X$. □

## 7. Trivial shape and pseudo-contractibility

Recall that a compact metric space $K$, is called *absolute neighbourhood retract*, written ANR, provided that whenever $K$ is embedded in a metric space $Y$, the embedded copy $K'$ of $K$ is a retract of some neighbourhood of $K'$ in $Y$. Let $X$ be a continuum. We say that $X$ has *trivial shape* provided that each mapping from $X$ into an ANR space is homotopic to a constant mapping.

It is well known the following result (see [4]).

**Theorem 43.** *Let $X$ be a continuum. The following sentences are equivalents:*

1. *$X$ has trivial shape.*
2. *$X$ can be written as $X = \bigcap_{n\in\mathbb{N}} X_n$, where $X_n$ is a contractible continuum and $X_{n+1} \subseteq X_n$ for every $n \in \mathbb{N}$.*
3. *$X$ can be written as an inverse limit of contractible continua.*
4. *For all $\varepsilon > 0$ there exists a contractible continuum $Y_\varepsilon$ and an $\varepsilon$-map $f_\varepsilon$ from $X$ onto $Y_\varepsilon$.*

The following proposition appears in [17].

**Proposition 44.** *Let $X$ be a compact metric space and let $Y$ be an ANR space. If $f$, $g : X \to Y$ are pseudo-homotopic, then they are homotopic.*

As an immediate consequence we obtain the following result.

**Proposition 45.** *Let $X$ be a compact metric space and let $Y$ be an ANR space. The space $X$ is pseudo-contractible with respect to $Y$ if and only if $X$ is contractible with respect to $Y$.*

**Proof.** Suppose that $X$ is pseudo-contractible with respect to $Y$. Let $f : X \to Y$ be a mapping, then by Definition 25, $f$ is pseudo-homotopic to a constant mapping. Since $Y$ is an ANR space, Proposition 44, shows that $f$ is homotopic to a constant mapping. Therefore, $X$ is contractible with respect to $Y$. The converse is trivial. □

Note that if $X$ is pseudo-contractible with respect to $Y$ and $Y$ is an ANR space (hence $X$ is contractible with respect to $Y$). Then, $Y$ is arcwise-connected if and only if $C(X, Y)$ is arcwise-connected.

**Corollary 46.** *Let $X$ be an ANR space. Then, $X$ is pseudo-contractible if and only if $X$ is contractible.*

**Corollary 47.** *Let $X$ be a compact metric space. Then, $X$ has trivial shape if and only if $X$ is pseudo-contractible with respect to each ANR space.*

**Proof.** Suppose that $X$ is pseudo-contractible with respect to each ANR space. By Proposition 45, $X$ is contractible with respect to each ANR space. Therefore $X$ has trivial shape. The converse is immediate. □

**Theorem 48.** *If $X$ is a pseudo-contractible continuum, then the following statements are true:*

1. *$X$ has trivial shape.*
2. *$X$ can be written as $X = \bigcap_{n\in\mathbb{N}} X_n$, where $X_n$ is a contractible continuum and $X_{n+1} \subseteq X_n$, for every $n \in \mathbb{N}$.*
3. *$X$ can be written as an inverse limit of contractible continua.*
4. *For all $\varepsilon > 0$, there exists a contractible continuum $Y_\varepsilon$ and an $\varepsilon$-map $f_\varepsilon$ from $X$ onto $Y_\varepsilon$.*

**Proof.** By Theorem 43, it is enough to prove that $X$ has trivial shape. Since $X$ is a pseudo-contractible continuum, then by Theorem 29, the space $X$ is pseudo-contractible respect to each compact metric space. In particular $X$ is a pseudo-contractible with respect to each ANR space. Therefore, by Corollary 47, $X$ has trivial shape. □

The converse of Theorem 48 is not true.

**Example 49.** The $\sin(\frac{1}{x})$ curve satisfies the conditions of Theorem 48, but it is not pseudo-contractible (see [5]).

On the other hand, notice that if $Y \approx^E_P \sin(\frac{1}{x})$ then $Y$ is not pseudo-contractible.

It is well known that $S^1$ does not have trivial shape. So, $S^1$ is not pseudo-contractible. Notice that whether $X \approx^E S^1$, then by Corollary 39, $X$ is not pseudo-contractible. The following continua are some examples of non pseudo-contractible continua because all of them have the same homotopy type that $S^1$.

1. The annulus $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 2)\}$.
2. The Solid Torus $S^1 \times D^2$.
3. The Möbius strip.

In general, if $X$ is of non-trivial shape and $Y \approx^E_P X$, then $Y$ is not pseudo-contractible.

Notice that $S^1 \not\approx^E W$, where $W$ denotes the Warsaw circle, but they have the same shape.

**Definition 50.** Let $X$ and $Y$ be topological spaces. A mapping $f : X \to Y$ is said to be *(pseudo-)essential* provided that $f$ is not (pseudo-)homotopic to any constant mapping of $X$ into $Y$. A mapping $f : X \to Y$ is said to be *(pseudo-)inessential* provided that $f$ is not (pseudo-)essential.

In this way, if $f : X \to Y$ is a (pseudo-)essential mapping, then $X$ is not (pseudo-)contractible with respect to $Y$. Note that whether $Y$ is an ANR space the notions of pseudo-essential mapping and essential mapping coincide.

The following theorem allows to construct more non pseudo-contractible continua.

**Theorem 51.** *If $X$ is a proper circle-like continuum, then $X$ is not pseudo-contractible.*

**Proof.** Since $X$ is a proper circle-like continuum, by [10, Theorem 3.2] there exists an essential mapping from $X$ onto $S^1$. Therefore, $X$ is not pseudo-contractible. □

In particular the pseudo-circle is not pseudo-contractible because it is a proper circle-like continuum.

## 8. Property b) and pseudo-contractibility

Recall that a space $X$ has *Property b)* provided that for each mapping $f : X \to S^1$, there exists a mapping $g : X \to \mathbb{R}$ such that $f = \exp \circ g$, where $\exp : \mathbb{R} \to S^1$ is defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$ and $\mathbb{R}$ denotes the real line. The mapping $g$ is called a *lift* of $f$.

The following result is well known.

**Theorem 52.** *[18, Theorem 6.2] Let $X$ be a compact metric space. The space $X$ is contractible with respect to $S^1$ if and only if $X$ has Property b).*

As a consequence of Proposition 45 and Theorem 52, we have the following.

**Corollary 53.** *Let $X$ be a compact metric space. The following conditions are equivalents:*

1. *$X$ is pseudo-contractible with respect to $S^1$.*
2. *$X$ is contractible with respect to $S^1$.*
3. *$X$ has Property b).*
4. *$C(X, S^1)$ is arcwise-connected.*

**Theorem 54.** *Let $X$ be a compact metric space. If $X$ is pseudo-contractible then $X$ has Property b).*

**Proof.** By Theorem 29, $X$ is pseudo-contractible with respect to $S^1$. Hence, by Corollary 53, $X$ has Property b). □

Note that if $X$ does not have Property $b$), then every space $Y$ such that $Y \approx^E X$ is not pseudo-contractible.

Recall that a continuum is said to be *unicoherent* provided that the intersection of any two of its subcontinua whose union is the whole continuum is connected. A continuum is said to be *hereditarily unicoherent* provided that each of its subcontinua is unicoherent. The following result is very important concerning unicoherence.

**Theorem 55.** *[18, Theorem 7.3] Every connected space $X$ having the property b) is unicoherent.*

**Corollary 56.** *Let $X$ be a continuum. If $X$ is pseudo-contractible then it is unicoherent.*

As a consequence we have that if $X$ is not unicoherent, then every space $Y$ such that $Y \approx^E_P X$ is not pseudo-contractible.

**Definition 57.** A continuum $X$ is *acyclic* if $\check{H}^1(X, \mathbb{Z}) = 0$; i.e., the first Čech cohomology group with integer coefficients is trivial.

By [6, Theorem 8.1], a continuum having Property b) is acyclic.

As a consequence we have the following result.

**Corollary 58.** *Let $X$ be a continuum. If $X$ is pseudo-contractible, then $X$ is acyclic.*

So, if $X$ is not acyclic, then every space $Y$ such that $Y \approx_P^E X$, is not pseudo-contractible.

A continuum is said to be a *dendroid*, if it is arcwise-connected and hereditarily unicoherent. A continuum is said to be *decomposable* provided that it can be written as the union of two proper subcontinua and it is called *hereditarily decomposable* if each of its nondegenerate subcontinua of is decomposable. A $\lambda$-*dendroid* means a hereditarily unicoherent and hereditarily decomposable continuum. It is well known that dendroids are $\lambda$-dendroids ([15, p. 226]). A *curve* means a one-dimensional continuum.

Finally, we will give some results when the topological space is a curve.

**Theorem 59.** *If $X$ is a pseudo-contractible curve then it is hereditarily unicoherent.*

**Proof.** By Theorem 48, $X$ has trivial shape. Thus, by [9, Theorem 2.1 (B)], $X$ is tree like. Therefore, by [2, Theorem 1], $X$ is hereditarily unicoherent. □

By using Theorem 59, we can give some examples of continua non pseudo-contractible. For instance:

1. The Menger sponge.
2. The Sierpinski carpet.
3. A Compactification of an arc with remainder a circle.

In general if $X$ is a non hereditarily unicoherent curve, then every space $Y$ such that $Y \approx_P^E X$ is not pseudo-contractible.

Notice that solenoids are not pseudo-contractible because they are hereditarily unicoherent, circle-like, and non acyclic curves.

It is known that every hereditarily decomposable continuum is a curve, hence we have the following result.

**Corollary 60.** *Let $X$ be a hereditarily decomposable continuum. If $X$ is pseudo-contractible, then $X$ is a $\lambda$-dendroid.*

The converse is not true, see Example 49.

A metric space $X$ is *homogeneous* provided that for each pair of points $x$, $y \in X$, there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.

**Theorem 61.** *If $X$ is a hereditarily decomposable pseudo-contractible continuum, then $X$ is not homogeneous.*

**Proof.** If we assume that $X$ is homogeneous, by [14, Lemma 5.2], there exists an essential mapping from $X$ onto $S^1$, a contradiction. □

In other words, every hereditarily decomposable homogeneous continuum is not pseudo-contractible or if there exists a homogeneous pseudo-contractible continuum it must be non hereditarily decomposable or equivalently the continuum must contains some indecomposable continuum.

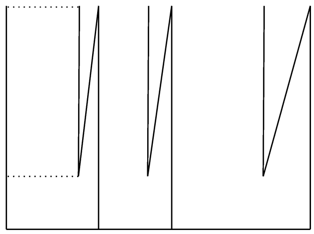

**Fig. 2.** Non pseudo-contractible dendroid.

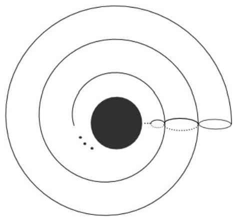

**Fig. 3.** Locally connected pseudo-contractible continuum.

**Theorem 62.** *If $X$ is a decomposable homogeneous curve then $X$ is not pseudo-contractible.*

**Proof.** If $X$ is a decomposable homogeneous curve, by [14, Lemma 5.1], there exists an essential mapping from $X$ onto $S^1$. Therefore, $X$ is not pseudo-contractible. □

As a consequence of the last result, we have that if $X$ is a decomposable pseudo-contractible curve, then $X$ is not homogeneous, or if there exists a homogeneous pseudo-contractible curve $X$, then $X$ is indecomposable. On the other hand, a continuum $X$ is said to be *hereditarily equivalent* if every subcontinuum nondegenerate of $X$ is homeomorphic to $X$. By [13, 2.6.39, Corollary] each hereditarily equivalent continuum is a curve. So, if there exists a pseudo-contractible homogeneous hereditarily equivalent continuum, it must be indecomposable.

Note that the circle of pseudo-arcs is not pseudo-contractible because it is a decomposable homogeneous curve.

**Corollary 63.** *Let $X$ be a curve. The following statements hold:*

1. *If $X$ is pseudo-contractible with arcwise-connected continuum as factor space, then $X$ is a uniformly arcwise-connected dendroid. Moreover, the dendroid $X$ is contractible.*
2. *If $X$ is pseudo-contractible and arcwise-connected then $X$ is a dendroid.*
3. *The space $X$ is locally connected and pseudo-contractible if and only if $X$ is a dendrite.*

The converses of Corollary 63.1 and Corollary 63.2 are not true. We consider the continuum given in [17, Corollary 5]. The continuum $X$ is a uniformly arcwise-connected dendroid but it is not pseudo-contractible. See Fig. 2.

Finally, there exists a locally connected continuum $X$ of dimension two, which is pseudo-contractible but is not contractible (W. Kuperberg, personal communication), see Fig. 3.

## 9. Questions

In this part we have some questions about pseudo-contractibility. First of all, notice that is natural to ask the following question.

**Question 64.** What kinds of continua satisfy that if $X$ is pseudo-contractible implies $X$ contractible?

Partial answers are given in Theorem 17, Corollary 46 and Corollary 63.

In particular and according to the second example given by W. Kuperberg, the next two questions were formulated.

**Question 65.** [3, Question 4.10] Is every pseudo-contractible dendroid, contractible?

**Question 66.** [12, Problem 118] Does there exist a curve which is pseudo-contractible but not contractible?

M. Sobolewski in [17] proves that the only non degenerate chainable pseudo-contractible continuum is the arc. In particular, the pseudo-arc and the Knaster-type indecomposable continua are not pseudo-contractible. In this sense is asked the following question.

**Question 67.** [16, Question 19] Does there exist a nondegenerate (hereditarily) indecomposable continuum which is pseudo-contractible?

Finally, concerning Remark 30, we have the following question.

**Question 68.** Does there exist a continuum $X$, which $C(X, X)$ is connected but $X$ is not pseudo-contractible?

## References


[1] D. Bellamy, A null pseudo-homotopic map onto a pseudo arc, Topol. Proc. 11 (1986) 1–5.
[2] J.H. Case, R.E. Chamberlin, Characterizations of tree-like continua, Pac. J. Math. 10 (1960) 73–84.
[3] J.J. Charatonik, Selected problems in continuum theory, Topol. Proc. 27 (1) (2003) 51–78.
[4] W.J. Charatonik, R.P. Roe, Inverse limits of continua having trivial shape, Houst. J. Math. 38 (2012) 1307–1312.
[5] W. Dębski, Pseudo-contractibility of the sin(1/x)-curve, Houst. J. Math. 20 (2) (1994) 365–367.
[6] C.H. Dowker, Mapping theorems for non-compact spaces, Am. J. Math. 69 (2) (1947) 200–242.
[7] A. Illanes, Pseudo-homotopies of the pseudo-arc, Comment. Math. Univ. Carol. 53 (4) (2012) 629–635.
[8] H. Katsuura, Pseudo-contraction and homotopy of the sin(1/x) curve, Proc. Am. Math. Soc. 115 (4) (1992) 1129–1138.
[9] J. Krasinkiewicz, Curves which are continuous images of tree-like continua are movable, Fundam. Math. 89 (3) (1975) 233–260.
[10] J. Krasinkiewicz, On the hyperspaces of snake-like and circle-like continua, Fundam. Math. 84 (1974) 155–164.
[11] K. Kuratowski, Topology, vol. II, Academic Press, New York, London, 1968.
[12] W. Lewis, Continuum theory problems, in: Proceedings of the 1983 Topology Conference, Houston, Tex., 1983, Topol. Proc. 8 (2) (1983) 361–394.
[13] S. Macías, Topics on Continua, Chapman and Hall/CRC, Boca Raton, FL, 2005.
[14] S. Macías, S.B. Nadler Jr., On hereditarily decomposable homogeneous continua, Topol. Proc. 33 (2009) 131–145.
[15] S.B. Nadler, Continuum Theory: An Introduction Monographs and Textbooks, Pure Appl. Math., vol. 158, Marcel Dekker, Inc., New York, 1992.
[16] E.M. Pearl (Ed.), Open Problems in Topology II, Elsevier, 2011.
[17] M. Sobolewski, Pseudo-contractibility of chainable continua, Topol. Appl. 154 (16) (2007) 2983–2987.
[18] G.T. Whyburn, Analytic topology, Colloq. Publ. – Am. Math. Soc. 28 (1942).