UNIVERSIDAD AUTÓNOMA DEL ESTADO DE MÉXICO

FACULTAD DE CIENCIAS

TESIS POR ARTÍCULOS ESPECIALIZADOS

TÍTULO:

FUNCIONES DE TAMAÑO FUERTE Y PROPIEDADES DECRECIENTES SECUENCIALES DE TAMAÑO FUERTE

QUE PARA OBTENER EL GRADO DE

DOCTOR EN CIENCIAS

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TOLUCA, MÉXICO, JUNIO 2019

Resumen

Un continuo es un espacio métrico compacto conexo y no vacío. Sean X un continuo y $C_n(X)$ la familia de subconjuntos de X, cerrados y no vacíos con a lo más n componentes. El espacio $C_n(X)$, llamdo n-ésimo hiperespacio de X, es considerado con la topología generada por la metrica de Hausdorff. Una función continua $\mu : C_n(X) \to [0,1]$ es de tamaño fuerte para $C_n(X)$ si

1. $\mu(A) = 0$ para todo $A \in F_n(X)$
2. $\mu(A) < \mu(B)$ si $A \subset B$, $A \neq B$ y $B \notin F_n(X)$

Una propiedad topológica $\mathcal{P}$ es llamada

1. *Propiedad de tamaño fuerte* si siempre que X tiene la propiedad $\mathcal{P}$, también la tiene todo nivel de tamaño fuerte.
2. *Propiedad decreciente secuencial de tamaño fuerte* si siempre que μ es una función de tamaño fuerte, $\{t_r\}_{r\in N}$ es una sucesión en el intervalo $(t_0, 1]$ tal que $t_r \to t$ y cada fibra $\mu^{-1}(t_r)$ tiene la propiedad $\mathcal{P}$, entonces $\mu^{-1}(t)$ tiene la propiedad $\mathcal{P}$.
3. *Propiedad creciente de tamaño fuerte* si simpre que μ es una función de tamaño fuerte y $t_0 \in [0, 1)$ tal que $\mu^{-1}(t_0)$ tiene la propiedad $\mathcal{P}$, entonces $\mu^{-1}(t)$ tiene la propiedad $\mathcal{P}$ para cada $t \in (t_0, 1)$.
4. Propiedad de Bloque de tamaño fuerte si siempre que X tiene la propiedad $\mathcal{P}$, todo bloque de tamaño fuerte también lo tiene.

Los resultados de la investigación se resumen en dos manuscritos. En el primero, que lleva por título "Sequential decreasing strong size properties", se muestra que ser localmente conexo, ser un continuo Kelley, la indescomponibilidad, la unicoherencia y encadenabilidad por continuos son propiedades decrecientes secuenciales de tamaño fuerte.

En el segundo, con nombre "Increasing strong size properties and block strong size properties", se prueba que la conexidad uniforme por trayectorias, la encadenabilidad uniforme por continuos y la conexidad local son propiedades crecientes de tamaño fuerte, adémas de mostrar algunas otras propiedades de bloques de tamaño fuerte.

Funciones de tamaño fuerte y propiedades decrecientes secuenciales de tamaño fuerte

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Resumen

Sean X un continuo y $C_n(X)$ la familia de subconjuntos de X, cerrados y no vacíos con a lo más n componentes. Decimos que $\mu : C_n(X) \to [0,1]$ es una función de tamaño fuerte para $C_n(X)$ si $\mu(A) = 0$ para todo $A \in F_n(X)$ y $\mu(A) < \mu(B)$ si $A \subset B$, $A \neq B$ y $B \notin F_n(X)$. Una propiedad topológica $\mathcal{P}$ es llamada propiedad decreciente secuencial de tamaño fuerte si μ es una función de tamaño fuerte, $\{t_r\}_{r\in\mathbb{N}}$ es una sucesión en el intervalo $(t_0, 1]$ tal que $t_r \to t$ y cada fibra $\mu^{-1}(t_r)$ tiene la propiedad $\mathcal{P}$, entonces $\mu^{-1}(t)$ tiene la propiedad $\mathcal{P}$. Una propiedad topológica $\mathcal{P}$ es llamada propiedad creciente de tamaño fuerte si μ es una función de tamaño fuerte y $t_0 \in [0,1)$ tal que $\mu^{-1}(t_0)$ tiene la propiedad $\mathcal{P}$, entonces $\mu^{-1}(t)$ tiene la propiedad $\mathcal{P}$ para cada $t \in (t_0, 1)$. El proyecto de investigación busca probar que ser localmente conexo, encadenable, de Kelley, atriodico, irreducible, unicoherente e indescomponible es una propiedad decreciente secuencial de tamaño fuerte y ser y que ser encadenable, ser el pseudoarco, ser un pseudosolenoide particular, ser pseudocirculo, ser hereditariamente indescomponible y tipo arco son propiedades crecientes de tamaño fuerte.

1. Introducción

Un continuo es un espacio métrico no degenerado, compacto y conexo. Para un continuo X y para un entero positivo n, denotamos por $C_n(X)$ al hiperespacio de los subconjuntos cerrados y no vacíos de X con a lo más n componentes, y por $F_n(X)$ el hiperespacio de los conjuntos finitos no vacíos con a lo más n puntos.. Ambos con la topología generada por la métrica de Hausdorff.(ver [4])

Una función de tamaño para $C_n(X)$ es una función continua $\mu : C_n(X) \to [0,1]$ tal que $\mu(\{x\}) = 0$ para todo $x \in X$ y $\mu(A) \leq \mu(B)$ si $A \subset B$ para todo $A, B \in C_n(X)$. Una función de Whitney es una función de tamaño que satisface que $\mu(A) < \mu(B)$ si $A \subset B$ y $A \neq B$ para todo $A, B \in C_n(X)$. Un nivel de Whitney es un subconjunto de la forma $\mu^{-1}(t)$ para una función de Whitney μ y $t \in [0, \mu(t)]$. Es conocido que los niveles de Whitney para $C(X)$ son subcontinuos de $C(X)$. Esto ultimo no siempre ocurre para funciones de Whitney para $C_n(X)$. (ver [2, p. 208])
Hiroshi Hosokawa en [1] introduce el concepto de funciones de tamaño fuerte, una generalización natural para las funciones de Whitney para $C_n(X)$. Una función $\mu : C_n(X) \to [0,1]$ es de tamaño fuerte si $\mu(A) = 0$ para todo $A \in F_n(X)$ y $\mu(A) < \mu(B)$ si $A \subset B$, $A \neq B$ y $B \notin F_n(X)$. Dichas funciones resultan ser monótonas es decir, los niveles de tamaño fuerte son conexos y así subcontinuos de $C_n(X)$.
Notemos que una función de tamaño fuerte restringida a $C(X)$ es una función de Whitney.

Sean X un continuo y μ una función de tamaño fuerte para $C_n(X)$. Una propiedad topológica $\mathcal{P}$ es llamada

- *propiedad de tamaño fuerte* si X tiene la propiedad $\mathcal{P}$, entonces todos los niveles la tienen.
- *propiedad decreciente secuencial de tamaño fuerte* si $\{t_r\}_{r\in\mathbb{N}}$ es una sucesión en el intervalo $(t_0, 1]$ tal que $t_r \to t$ y cada fibra $\mu^{-1}(t_r)$ tiene la propiedad $\mathcal{P}$, entonces $\mu^{-1}(t)$ tiene la propiedad $\mathcal{P}$
- *propiedad creciente de tamaño fuerte* si $t_0 \in [0,1)$ tal que $\mu^{-1}(t_0)$ tiene la propiedad $\mathcal{P}$, entonces $\mu^{-1}(t)$ tiene la propiedad $\mathcal{P}$ para cada $t \in (t_0, 1)$

Un problema general en Teoría de continuos y sus hiperespacios es determinar si una propiedad topológica es de Whitney, decreciente secuencial de Whitney o creciente de Whitney. El propósito de la investigación es generalizar estos resultados para las funciones de tamaño fuerte.

2. Antecedentes

Muchos autores han estudiado las propiedades de tamaño fuerte para $C_n(X)$ con $n = 1$. En el Capitulo $VIII$ de [2] se presenta una detallada lista de resultados sobre este tema. En 2010 Hosokawa en [1] prueba que ser localmente conexo, ser arcoconexo y la aposindesis son propiedades de tamaño fuerte para $C_n(X)$ con $n \geq 1$. En 2013 César Piceno y Sergio Macías en [3] demuestran que ser finitamente aposindético, numerablemente aposindético, encadenable y aciclico para localmente conexos son propiedades de tamaño fuerte para $C_n(X)$.

Por otro lado, Fernando Orozco en [5] y [6] prueba que ser localmente conexo, encadenable, de Kelley, atriodico, irreducible, unicoherente e indescomponible son propiedades decrecientes secuenciales de tamaño fuerte para $C_n(X)$ con $n = 1$. Además en 2010 en [7], prueba también que ser encadenable, ser el pseudoarco, ser un pseudosolenoide particular, ser pseudocirculo, ser hereditariamente indescomponible y tipo arco son propiedades crecientes de tamaño fuerte para $C_n(X)$ y $n = 1$.

3. Problema

1. Encontrar propiedades topológicas que sean propiedades decrecientes secuenciales de tamaño fuerte para $C_n(X)$ con $n \geq 1$.
2. Encontrar propiedades topológicas que sean propiedades crecientes de tamaño fuerte para $C_n(X)$ con $n \geq 1$.
3. Encontrar propiedades generales de las funciones de tamaño fuerte.

4. Objetivos y Metas

- Resolver el problema 1 para las siguientes propiedades: ser localmente conexo, ser encadenable, ser de Kelley, ser atriodico, ser irreducible, ser unicoherente y ser indescomponible.
- Resolver el problema 2 para las siguientes propiedades: ser encadenable, ser el pseudoarco, ser un pseudosolenoide particular, ser pseudocirculo, ser hereditariamente indescomponible y tipo arco.
- Dar respuestas parciales para el problema 3.
- Publicar 2 artículos que contemplen los resultados de la investigación.

5. Metodología

Reuniones semanales con el grupo de investigadores enlistado en la Sección para discutir los avances obtenidos y proponer nuevas líneas de investigación.

Revisar de forma exhaustiva la bibliografía existente para generar nuevas líneas de investigación.

Participar semanalmente en el Seminario de Hiperespacios de Continuos de la Facultad de Ciencias de la UAEMéx.

Participar en el seminario de Hiperespacios que dirige el Dr. Alejandro Illanes Mejía en el Instituto de Matemáticas de la UNAM.

Crear grupos de trabajo con estudiantes de la licenciatura de matemáticas de la Facultad de Ciencias de la UAEMéx.

Realizar anualmente una estancia de investigación con investigadores de prestigio internacional.

Invitar anualmente a un investigador de prestigio internacional a la Facultad de Ciencias de la UAEMéx a participar con el grupo de trabajo.

Difundir anualmente los resultados y avances parciales obtenidos en la investigación en congresos nacionales e internacionales.

DE GRUYTER

Mathematica Slovaca

DOI: 10.1515/ms-2017-0176
Math. Slovaca **68** (2018), No. 5, 1141–1148

SEQUENTIAL DECREASING STRONG SIZE PROPERTIES

Félix Capulín — Miguel A. Lara — Fernando Orozco-Zitli

(*Communicated by David Buhagiar*)

ABSTRACT. Let X be a continuum. The n-fold hyperspace $C_n(X)$, $n < \infty$, is the space of all nonempty closed subsets of X with at most n components. A topological property $\mathcal{P}$ is said to be a (an almost) sequential decreasing strong size property provided that if μ is a strong size map for $C_n(X)$, $\{t_j\}_{j=1}^{\infty}$ is a sequence in the interval $(t,1)$ such that $\lim t_j = t \in [0,1)$ $(t \in (0,1))$ and each fiber $\mu^{-1}(t_j)$ has property $\mathcal{P}$, then so does $\mu^{-1}(t)$. In this paper we show that the following properties are sequential decreasing strong size properties: being a Kelley continuum, local connectedness, continuum chainability and, unicoherence. Also we prove that indecomposability is an almost sequential decreasing strong size property.

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1. Introduction

In [12] and [13] F. Orozco-Zitli proved that atriodicity, containing no arc, irreducibility, indecomposability, being a Kelley continuum, local connectedness, continuum chainability and unicoherence are sequential decreasing Whitney properties. Sequential decreasing strong size properties are the natural generalization of sequential decreasing Whitney properties. We prove that being a Kelley continuum, local connectedness, continuum chainability and unicoherence are sequential decreasing strong size properties. Also we prove that indecomposibility is an almost sequential decreasing strong size property.

2. Preliminaries

Given a metric space (Z,d) and a subset B of Z. If $x \in Z$ and $\varepsilon > 0$, let $\mathcal{V}_{\varepsilon}^{d}(x) = \{y \in X : d(x,y) < \varepsilon\}$ and $N(\varepsilon, B) = \bigcup\{\mathcal{V}_{\varepsilon}^{d}(x) : x \in B\}$. We denote by $\mathrm{cl}(B)$ the closure of B in Z. Further, $\mathrm{diam}(B)$ will denote the diameter of B. A *continuum* is a nonempty compact, connected, metric space. A *subcontinuum* of a space Z is a continuum contained in Z.

The symbol $\mathbb{N}$ denotes the set of positive integers. Let X be a continuum. For each $n \in \mathbb{N}$, $C_n(X)$ denotes the hyperspace of all nonempty closed subsets of X with at most n components; $C_n(X)$ is called the *n-fold hyperspace* of X (thus, $C_1(X)$ is the classical hyperspace of all subcontinua of X and, as is customary, is denoted by $C(X)$ instead of $C_1(X)$). The symbol $F_n(X)$ denotes the *n-fold symmetric product* of a continuum X; that is, $F_n(X) = \{A \in C_n(X) : \text{A has at most } n \text{ points}\}$.

2010 Mathematics Subject Classification: Primary 54C05, 54C10, 54B20; Secondary 54B15.
Keywords: n-fold hyperspace, strong size property, strong size map, Kelley continuum, indecomposability, local connectedness, continuum chainability and unicoherence.

We topologize these sets with the Hausdorff metric H, defined as follows: $H(A,B) = \inf\{\varepsilon > 0 : A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\}$, (see [10: p. 1]). We denote by H^2 the corresponding Hausdorff metric for $C(C_n(X))$. An *order arc* in $C_n(X)$ is an arc $\alpha\colon [0,1] \to C_n(X)$ such that if $0 \le s < t \le 1$, then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \ne \alpha(t)$.

A *map* means a continuous function. A *size map* for $C_n(X)$ is a map $\omega\colon C_n(X) \to [0,1]$ such that $\omega(\{x\}) = 0$ for each $x \in X$ and $\omega(A) \le \omega(B)$ if $A \subset B$ for each $A, B \in C_n(X)$. A *strong size map* for $C_n(X)$ is a map $\mu\colon C_n(X) \to [0,1]$ such that

(i) $\mu(A) = 0$ for each $A \in F_n(X)$,

(ii) if $A \subset B$, $A \ne B$ and $B \notin F_n(X)$, then $\mu(A) < \mu(B)$

(iii) $\mu(X) = 1$ (see [2: p. 956]).

By Theorem 2.10 of [2: p. 958], every strong size map is monotone. Each set of the form $\mu^{-1}(t)$ for any strong size map for $C_n(X)$ and any $t \in [0,1]$ is called a *strong size level* of $C_n(X)$.

Let X be a continuum and let μ be a strong size map for $C_n(X)$. Let $A \in C_n(X)$. If $t \in [0, \mu(A))$, let $C(A,t) = \{B \in \mu^{-1}(t) : B \subset A \text{ and each component of } A \text{ intersects } B\}$. Also, if $t \in [\mu(A), 1)$, let $C_A^t = \{B \in \mu^{-1}(t) : A \subset B \text{ and each component of } B \text{ intersects } A\}$. Notice that if $t \in [\mu(A), 1)$, then C_A^t is closed in $\mu^{-1}(t)$. If $t \in [0, \mu(A))$, then $C(A,t)$ is closed in $\mu^{-1}(t)$. Then, for each $t \in [0, \mu(A))$, $C(A,t)$ is a subcontinuum of $\mu^{-1}(t)$ (see [2: Theorem 2.14, p. 959]).

A topological property $\mathcal{P}$ is said to be a *sequential decreasing strong size property* provided that if μ is a strong size map for $C_n(X)$, $t \in [0,1)$, $\{t_j\}_{j\in\mathbb{N}}$ is a sequence into the interval $(t,1)$ such that $\lim t_j = t$ and each fiber $\mu^{-1}(t_j)$ has property $\mathcal{P}$, then so does $\mu^{-1}(t)$.

A topological property $\mathcal{P}$ is said to be an *almost sequential decreasing strong size property* provided that if μ is a strong size map for $C_n(X)$, $t \in (0,1)$, $\{t_j\}_{j\in\mathbb{N}}$ is a sequence into the interval $(t,1)$ such that $\lim t_j = t$ and each fiber $\mu^{-1}(t_j)$ has property $\mathcal{P}$, then so does $\mu^{-1}(t)$.

Let $\sigma\colon C(C_n(X)) \to C_n(X)$ be a function given by $\sigma(\mathcal{A}) = \bigcup\{A : A \in \mathcal{A}\}$, by [3: p. 23], σ is a map and, by [6: Lemma 7.2, p. 250]) it is well defined; it is clear that σ is onto. The map σ is called the union map.

A continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua. A continuum which is not decomposable is said to be *indecomposable*.

A continuum X is said to be *unicoherent* provided that whenever A and B are subcontinua of X such that $A \cup B = X$, then $A \cap B$ is connected.

A continuum X is called a *Kelley continuum* provided that given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $p, q \in X$ with $d(p,q) < \delta$ and $p \in A \in C(X)$, then there exists $B \in C(X)$ such that $q \in B$ and $H(A,B) < \varepsilon$.

A continuum X is *continuum chainable* if for each $\varepsilon > 0$ and each pair of points $p \ne q$ in X, there is a finite sequence of subcontinua $\{C_1, \dots, C_r\}$ of X such that $\operatorname{diam}(C_i) < \epsilon$, $p \in C_1$, $q \in C_r$ and $C_i \cap C_{i+1} \ne \emptyset$ for every $i \le r-1$.

Remark 2.1. It can easily be proved that a continuum X is a Kelley continuum if and only if for every point $p \in X$ and for each $\varepsilon > 0$, there exists $\delta > 0$ with the property that if $A \in C(X)$, $p \in A$ and $q \in \mathcal{V}_\delta^d(p)$, then there exists $B \in C(X)$ such that $q \in B$ and $H(A,B) < \varepsilon$.

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3. Preliminary results

Lemma 3.1. *Let X be a continuum. Let $\{A_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ be sequences of $C_n(X)$ such that $\lim A_k = A$ and $\lim B_k = B$. If and each component of B_k intersects A_k for each $k \in \mathbb{N}$, then each component of B intersects A.*

Proof. Let C be a component of B and let $x \in C$. Then there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $\lim x_k = x$ and $x_k \in B_k$ for each $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, let C_k be the component of B_k such that $x_k \in C_k$. Since $\{C_k\}_{k\in\mathbb{N}}$ is a sequence of elements of $C(X)$, by the compactness of $C(X)$ we may assume that $\{C_k\}_{k\in\mathbb{N}}$ converges to some element D of $C(X)$. Notice that $D \subset C$. Since $A_k \cap C_k \neq \emptyset$ for each $k \in \mathbb{N}$, $A \cap D \neq \emptyset$. Hence $A \cap C \neq \emptyset$. Therefore, every component of B intersects A. □

Lemma 3.2. *Let μ be a strong size map for $C_n(X)$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $A, B \in C_n(X)$ satisfy that each component of B intersects A, $A \subset N(\delta, B)$ and $|\mu(A) - \mu(B)| < \delta$, then $H(A, B) < \varepsilon$.*

Proof. Suppose that the lemma is false for some $\varepsilon > 0$. Then there are two sequences $\{A_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ in $C_n(X)$ such that, for each $m \in \mathbb{N}$, $A_m \subset N(\frac{1}{m}, B_m)$, each component of B_m intersects A_m, $|\mu(A_m) - \mu(B_m)| < \frac{1}{m}$ and $H(A_m, B_m) \geq \varepsilon$. We assume, without loss of generality, that $\lim A_k = A$ for some $A \in C_n(X)$ and $\lim B_k = B$ for some $B \in C_n(X)$. Notice that $A \subset B$. We will prove that $A = B$. If $B \in F_n(X)$, by Lemma 3.1, $A = B$. Now if $B \notin F_n(X)$, by the continuity of μ, $\mu(A) = \mu(B)$. Thus, $A = B$. Since $\lim B_k = B = A$ and $\lim A_k = A$, there exists $m \in \mathbb{N}$ such that $H(A_m, B_m) \leq H(A_m, A) + H(B_m, A) < \varepsilon$, a contradiction. □

Lemma 3.3. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0, 1)$. If $t \in (t_0, 1)$ and $\mathcal{A} \in C(\mu^{-1}(t))$, then $\bigcup\{C(A, t_0) : A \in \mathcal{A}\}$ is a subcontinuum of $\mu^{-1}(t_0)$.*

Proof. Let $\mathfrak{B} = \bigcup\{C(A, t_0) : A \in \mathcal{A}\}$. We will prove that $\mathfrak{B}$ is closed. Let $\{B_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$ such that $\lim B_k = B$ for some $B \in C_n(X)$. Then, there exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that, for each $k \in \mathbb{N}$, $B_k \in C(A_k, t_0)$. Since $\mathcal{A}$ is compact, we may assume that $\lim A_k = A$ for some $A \in \mathcal{A}$. Then, $B \subset A$ and $B \in \mu^{-1}(t_0)$. By Lemma 3.1, each component of A intersects B. Thus, $B \in C(A, t_0)$. Hence $B \in \mathfrak{B}$.

On the other hand, suppose that $\mathfrak{B}$ is not connected. Then, there are two nonempty disjoint closed subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathfrak{B}$ such that $\mathfrak{B} = \mathcal{L}_1 \cup \mathcal{L}_2$.

For each $i \in \{1, 2\}$, let $\mathcal{L}_i^* = \{A \in \mathcal{A} : C(A, t_0) \subset \mathcal{L}_i\}$. Notice that $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ are nonempty disjoint subsets of $\mathcal{A}$ and $\mathcal{L}_1^* \cup \mathcal{L}_2^* = \mathcal{A}$. Let $i \in \{1, 2\}$. In order to prove that $\mathcal{L}_i^*$ is closed, let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{L}_i^*$ converging to an element $A \in \mathcal{A}$. Since $\{C(A_k, t_0)\}_{k\in\mathbb{N}}$ is a sequence of elements of $C(\mu^{-1}(t_0))$. By compactness we may assume that the sequence $\{C(A_k, t_0)\}_{k\in\mathbb{N}}$ converges to an element $\mathcal{D} \in C(\mu^{-1}(t_0))$. Thus, since $\bigcup\limits_{k\in\mathbb{N}} C(A_k, t_0) \subset \mathcal{L}_i$ and $\mathcal{L}_i$ is closed, $\mathcal{D} \subset \mathcal{L}_i$. Now, we need to show that $\mathcal{D} \subset C(A, t_0)$. Let $B \in \mathcal{D}$. Then, there exists a sequence $\{B_k\}_{k\in\mathbb{N}}$ in $\mathfrak{B}$ such that, for each $k \in \mathbb{N}$, $B_k \in C(A_k, t_0)$ and $\lim B_k = B$. Then, $B \subset A$ and $B \in \mu^{-1}(t_0)$. By Lemma 3.1, $B \in C(A, t_0)$. We have shown that $\mathcal{D} \subset C(A, t_0)$. Thus, since $C(A, t_0)$ is connected, $C(A, t_0) \subset \mathcal{L}_i$. Hence $A \in \mathcal{L}_i^*$ and $\mathcal{L}_i^*$ is closed. Therefore, $\mathcal{A}$ is not connected, a contradiction. This completes the proof that $\mathfrak{B}$ is a subcontinuum of $\mu^{-1}(t_0)$. □

The proof of the following lemma is similar to the one given for Lemma 3.2 of [8: p. 106] (see [10: Lemma 14.8.1, p. 406]).

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Lemma 3.4. *Let μ be a strong size map for $C_n(X)$. If $A \in C_n(X)$ and $t \in (\mu(A), 1)$, then C^t_A is arcwise connected.*

Lemma 3.5. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $t \in (t_0, 1]$ and $\mathcal{A} \in C(\mu^{-1}(t_0))$, then $\bigcup\{C^t_A : A \in \mathcal{A}\}$ is a subcontinuum of $\mu^{-1}(t)$.*

Proof. Let $\mathcal{S} = \bigcup\{C^t_A : A \in \mathcal{A}\}$. Using similar ideas as in Lemma 3.3 we can prove that $\mathcal{S}$ is closed in $\mu^{-1}(t)$. Now suppose $\mathcal{S}$ is not connected. Then, there exist two nonempty disjoint closed subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{S}$ such that $\mathcal{S} = \mathcal{F}_1 \cup \mathcal{F}_2$. For each $i \in \{1,2\}$, let $\mathcal{L}^*_i = \{A \in \mathcal{A} : C^t_A \subset \mathcal{F}_i\}$. Notice that $\mathcal{L}^*_1$ and $\mathcal{L}^*_2$ are nonempty disjoint subsets of $\mathcal{A}$ and $\mathcal{L}^*_1 \cup \mathcal{L}^*_2 = \mathcal{A}$. Let $i \in \{1,2\}$. In order to prove that $\mathcal{L}^*_i$ is closed, we consider a sequence $\{A_k\}_{k\in\mathbb{N}}$ in $\mathcal{L}^*_i$ converging to an element $A \in \mathcal{A}$. Since $\{C^t_{A_k}\}_{k\in\mathbb{N}}$ is a sequence of elements of $C(\mu^{-1}(t))$. By compactness we may assume that $\{C^t_{A_k}\}_{k\in\mathbb{N}}$ converges to an element $\mathcal{D} \in C(\mu^{-1}(t))$. Thus, since $\bigcup_{k\in\mathbb{N}} C^t_{A_k} \subset \mathcal{F}_i$ and $\mathcal{F}_i$ is closed, $\mathcal{D} \subset \mathcal{F}_i$. Now, we need to show that $\mathcal{D} \subset C^t_A$. Let $B \in \mathcal{D}$. Then there exists a sequence $\{B_k\}_{k\in\mathbb{N}}$ in $\mathcal{S}$ such that, for each $k \in \mathbb{N}$, $B_k \in C^t_{A_k}$ and $\lim B_k = B$. Then $A \subset B$ and $B \in \mu^{-1}(t)$. By Lemma 3.1, $B \in C^t_A$. We have shown that $\mathcal{D} \subset C^t_A$. Thus, since C^t_A is connected (see Lemma 3.4), $C^t_A \subset \mathcal{F}_i$. Hence $A \in \mathcal{L}^*_i$ and $\mathcal{L}^*_i$ is closed. Therefore, $\mathcal{A}$ is not connected, a contradiction.
This completes the proof that $\mathcal{S}$ is a subcontinuum of $\mu^{-1}(t)$. □

For the following, it is known that if $\mathcal{A} \in C(C_n(X))$, then $\sigma(\mathcal{A}) \in C_n(X)$, see [7: Lemma 7.2].

Lemma 3.6. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $\mathcal{A}$ is a nondegenerate subcontinuum of $\mu^{-1}(t_0)$ and $t \in [t_0, \mu(\sigma(\mathcal{A})))$, then $X(\mathcal{A}, t) = \{B \in \mu^{-1}(t)$: there exists a subcontinuum $\mathcal{B}$ of $\mathcal{A}$ such that $\sigma(\mathcal{B}) = B\}$ is a subcontinuum of $\mu^{-1}(t)$.*

Proof. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(s) = s - t_0$. Clearly, f is a homeomorphism. Define $\omega\colon C(\mathcal{A}) \to \mathbb{R}$ by $\omega(\mathfrak{B}) = f(\mu(\sigma(\mathfrak{B})))$. Notice that:

(1) ω is a map;

(2) $\omega(\{D\}) = 0$ for each $D \in \mathcal{A}$;

(3) if $\mathfrak{B}_1, \mathfrak{B}_2 \in C(\mathcal{A})$, with $\mathfrak{B}_1 \subset \mathfrak{B}_2$, then $\omega(\mathfrak{B}_1) \le \omega(\mathfrak{B}_2)$.

Thus, ω is a size map for $C(\mathcal{A})$. Hence for each $t \in [t_0, \mu(\sigma(\mathcal{A}))]$, $\omega^{-1}(f(t)) = \{\mathfrak{B} \in C(\mathcal{A}) : \mu(\sigma(\mathfrak{B})) = t\}$ is a subcontinuum of $C(\mathcal{A})$ (see [11: p. 243]). Since $X(\mathcal{A}, t) = \sigma(\omega^{-1}(f(t)))$ and σ is continuous, $X(\mathcal{A}, t)$ is a subcontinuum of $\mu^{-1}(t)$. □

Lemma 3.7. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $A \in \mu^{-1}(t_0)$ and $r \in (t_0, 1)$, then there exists a subcontinuum $\mathcal{A}$ of $\mu^{-1}(t_0)$ such that $A \in \mathcal{A}$ and $\mu(\sigma(\mathcal{A})) = r$.*

Proof. Let $\alpha\colon [0,1] \to C(\mu^{-1}(t_0))$ be an order arc such that $\alpha(0) = \{A\}$ and $\alpha(1) = \mu^{-1}(t_0)$. Since the composition $\mu \circ \sigma \circ \alpha$ is continuous, and $\mu(\sigma(\alpha(0))) = t_0$ and $\mu(\sigma(\alpha(1))) = 1$, there exists $s \in (0,1)$ such that $\mu(\sigma(\alpha(s))) = r$. Note that $\alpha(s) \in C(C_n(X))$ because $C(\mu^{-1}(t_0)) \subset C(C_n(X))$. Clearly, $\mathcal{A} = \alpha(s)$ has the required properties, and the lemma is proved. □

Lemma 3.8. *Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $A, B \in \mu^{-1}(t_0)$ and $A \neq B$, then there exists $s \in (t_0, 1)$ such that if $\mathcal{A}, \mathcal{B} \in C(\mu^{-1}(t_0))$, $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\mu(\sigma(\mathcal{A}))$, $\mu(\sigma(\mathcal{B})) \in (t_0, s)$, then $\sigma(\mathcal{A}) \neq \sigma(\mathcal{B})$.*

Proof. Let $a \in A \setminus B$ and let $\varepsilon > 0$ be such that $\mathcal{V}^d_\varepsilon(a) \cap B = \emptyset$. Let $\delta > 0$ be as in Lemma 3.2 for the number ε. Let $s = \min\{t_0 + \delta, 1\}$. Let $\mathcal{A}$ and $\mathcal{B}$ two subcontinua of $\mu^{-1}(t_0)$ such that $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\mu(\sigma(\mathcal{A}))$, $\mu(\sigma(\mathcal{B})) \in (t_0, s)$. Since $\mu(\sigma(\mathcal{B})) - \mu(B) < \delta$, $B \subset \sigma(\mathcal{B})$ and each component of $\sigma(\mathcal{B})$ intersects B (see [1: Lemma 3.1, p. 241]), by the choice of δ, $H(B, \sigma(\mathcal{B})) < \varepsilon$. Thus $\sigma(\mathcal{B}) \subset N(\epsilon, B)$. Therefore, $A \not\subseteq \sigma(\mathcal{B})$ and $\sigma(\mathcal{A}) \neq \sigma(\mathcal{B})$ □

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4. Main results

Theorem 4.1. *Local connectedness is a sequential decreasing strong size property.*

Proof. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. If $\{t_j\}_{j\in\mathbb{N}}$ is a sequence in $(t_0,1]$ converging to t_0 and each fiber $\mu^{-1}(t_j)$ is locally connected, we will prove that $\mu^{-1}(t_0)$ is locally connected. Let $\varepsilon > 0$. Let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon}{4}$. Let $t_J \in (t_0, t_0+\delta)$. Since $\mu^{-1}(t_J)$ is locally connected, by [14: 15.7, p. 23], there exists a finite set $\{\mathcal{A}_1,\ldots,\mathcal{A}_m\}$ of subcontinua of $\mu^{-1}(t_J)$ such that $\operatorname{diam}(\mathcal{A}_i) < \frac{\varepsilon}{4}$ for each $i \le m$, and $\mu^{-1}(t_J) = \bigcup_{i=1}^{m} \mathcal{A}_i$. For each $i \in \{1,\ldots,m\}$, define $\mathfrak{B}_i = \bigcup\{C(A,t_0) : A \in \mathcal{A}_i\}$.

Now we will prove that $\mu^{-1}(t_0) = \bigcup_{i=1}^{m} \mathfrak{B}_i$. Notice that by Lemma 3.3, for each $i \le m$, $\mathfrak{B}_i$ is a subcontinuum of $\mu^{-1}(t_0)$. On the other hand if $D \in \mu^{-1}(t_0)$, there exists an order arc $\alpha\colon [0,1] \to C_n(X)$ such that $\alpha(0) = D$ and $\alpha(1) = X$. Since $\mu\circ\alpha\colon [0,1]\to[0,1]$ is a mapping, there exists $s \in (0,1)$ such that $\mu(\alpha(s)) = t_J$. Notice that $\alpha(s) \in \mathcal{A}_i$ for some $i \in \{1,\ldots,m\}$ and $\alpha(0) \subset \alpha(s)$ by definition of order arc. So, $D \in C(\alpha(s), t_0, n) \subset \mathcal{B}_i$. Thus $\mu^{-1}(t_0) = \bigcup_{i=1}^{m} \mathfrak{B}_i$. Finally we will show that $\operatorname{diam}(\mathfrak{B}_i) < \varepsilon$. Let $i \le m$. Consider $B \in \mathfrak{B}_i$ and $A \in \mathcal{A}_i$, such that $B \in C(A, t_0, n)$. Notice that $|\mu(A) - \mu(B)| < \delta$. So, by the choice of δ, $H(A,B) < \frac{\varepsilon}{4}$. Since $\operatorname{diam}(\mathcal{A}_i) < \frac{\varepsilon}{4}$, $H(M,B) < \frac{\varepsilon}{2}$ for each $M \in \mathcal{A}_i$. Therefore, $\operatorname{diam}(\mathfrak{B}_i) < \varepsilon$ and by [14: 15.7, p. 23], $\mu^{-1}(t_0)$ is locally connected. □

Theorem 4.2. *Continuum chainability is a sequential decreasing strong size property.*

Proof. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. Suppose that $\{t_j\}_{j\in\mathbb{N}} \subset (t_0,1]$ is a sequence which converges to t_0 and each fiber $\mu^{-1}(t_j)$ is continuum chainable.

In order to prove that $\mu^{-1}(t_0)$ is continuum chainable, let $A_1 \ne A_2 \in \mu^{-1}(t_0)$. Let $\varepsilon > 0$ and let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon}{4}$. For A_1 and A_2, let $s \in (t_0,1)$ be as in Lemma 3.8. Let $t_J \in (t_0, \min\{t_0+\delta, s\})$. By Lemma 3.7, for each $k \in \{1,2\}$, there exists $\mathcal{M}_k \in C(\mu^{-1}(t_0))$ such that $\mu(\sigma(\mathcal{M}_k)) = t_J$ and $A_k \in \mathcal{M}_k$. By [1: Lemma 3.1, p. 241], $A_k \in C(\sigma(\mathcal{M}_k), t_0)$ for each $k \in \{1,2\}$. By the choice of s, $\sigma(\mathcal{M}_1) \ne \sigma(\mathcal{M}_2)$. Since $\mu^{-1}(t_J)$ is continuum chainable, there exists a finite sequence $\{\mathcal{A}_1,\ldots,\mathcal{A}_m\}$ of subcontinua of $\mu^{-1}(t_J)$ such that $\sigma(\mathcal{M}_1) \in \mathcal{A}_1$, $\sigma(\mathcal{M}_2) \in \mathcal{A}_m$, $\mathcal{A}_i \cap \mathcal{A}_{i+1} \ne \emptyset$ for each $i < m$ and $\operatorname{diam}(\mathcal{A}_i) < \frac{\varepsilon}{4}$, for each $i \le m$. By Lemma 3.3, $\mathfrak{B}_i = \bigcup\{C(D,t_0) : D \in \mathcal{A}_i\}$ is a subcontinuum of $\mu^{-1}(t_0)$, for each $i \in \{1,\ldots,m\}$. Clearly, $A_1 \in \mathfrak{B}_1$, $A_2 \in \mathfrak{B}_m$ and $\mathfrak{B}_i \cap \mathfrak{B}_{i+1} \ne \emptyset$ for each $i < m$. Let $i \le m$. Now we show that $\operatorname{diam}(\mathfrak{B}_i) < \varepsilon$. Let $D \in \mathfrak{B}_i$. We consider $G \in \mathcal{A}_i$ such that $D \in C(G,t_0)$. Since $\mu(G) - \mu(D) < \delta$, by the choice of δ, $H(D,G) < \frac{\varepsilon}{4}$. So, since $\operatorname{diam}(\mathcal{A}_i) < \frac{\varepsilon}{4}$, $H(M,D) < \frac{\varepsilon}{2}$ for each $M \in \mathcal{A}_i$. Hence $\operatorname{diam}(\mathfrak{B}_i) < \varepsilon$. Since $\mathcal{A}_i \cap \mathcal{A}_{i+1} \ne \emptyset$ for each $i < m$, $\mathcal{B}_i \cap \mathcal{B}_{i+1} \ne \emptyset$ for each $i < m$. Therefore, $\mu^{-1}(t_0)$ is continuum chainable. □

Theorem 4.3. *The property of being a Kelley continuum is a sequential decreasing strong size property.*

Proof. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0,1)$. Suppose that $\{t_j\}_{j\in\mathbb{N}} \subset (t_0,1]$ is a sequence converging t_0 and each fiber $\mu^{-1}(t_j)$ is a Kelley continuum.

We will prove that $\mu^{-1}(t_0)$ is a Kelley continuum. Suppose that the theorem is false for some $P \in \mu^{-1}(t_0)$ and some $\varepsilon > 0$. By Remark 2.1, there are two sequences $\{\mathcal{A}_m\}_{m\in\mathbb{N}} \subset C(\mu^{-1}(t_0))$ and $\{Q_m\}_{m\in\mathbb{N}} \subset \mu^{-1}(t_0)$ such that, for each $m \in \mathbb{N}$, $P \in \mathcal{A}_m$, $H(P,Q_m) < \frac{1}{m}$, and if $Q_m \in \mathcal{G} \in C(\mu^{-1}(t_0))$, $H^2(\mathcal{A}_m,\mathcal{G}) \ge \varepsilon$. Let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon}{12}$.

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Let $t_J \in (t_0, t_0 + \delta)$. By Lemma 3.7, for each $m \in \mathbb{N}$, there exists $\mathcal{D}_m \in C(\mu^{-1}(t_0))$ such that $Q_m \in \mathcal{D}_m$ and $\mu(\sigma(\mathcal{D}_m)) = t_J$. We may assume that $\lim \mathcal{A}_m = \mathcal{A}$ and $\lim \mathcal{D}_m = \mathcal{D}$ for some $\mathcal{A}, \mathcal{D} \in C(\mu^{-1}(t_0))$. Since $\lim Q_m = P \in \mathcal{A}$ and $Q_m \in \mathcal{D}_m$ for each $m \in \mathbb{N}$, we have $P \in \mathcal{D}$. Thus, $P \in \mathcal{D} \cap \mathcal{A}$ and therefore, $\mathcal{A} \cup \mathcal{D} \in C(\mu^{-1}(t_0))$.

We prove that $H^2(\mathcal{A}, \mathcal{A} \cup \mathcal{D}) < \frac{\varepsilon}{6}$, . To this end, it is enough to prove that $\operatorname{diam}(\mathcal{D}) < \frac{\varepsilon}{6}$ proving that $H(\sigma(\mathcal{D}), E) < \frac{\varepsilon}{12}$ for every $E \in \mathcal{D}$. Let $E \in \mathcal{D}$, by [1: Lemma 3.1, p. 241], $E \in C(\sigma(\mathcal{D}), t_0)$. Since $|\mu(\sigma(\mathcal{D})) - \mu(E)| = |t_J - t_0| < \delta$. By the choice of δ, $H(\sigma(\mathcal{D}), E) < \frac{\varepsilon}{12}$. Thus, $\operatorname{diam}(\mathcal{D}) < \frac{\varepsilon}{6}$. Hence $H^2(\mathcal{A}, \mathcal{A} \cup \mathcal{D}) < \frac{\varepsilon}{6}$.

Notice that $\sigma(\mathcal{D}) \in X(\mathcal{A} \cup \mathcal{D}, t_J)$. Since $\mu^{-1}(t_J)$ is a Kelley continuum, there exists $\eta > 0$ such that if $L \in \mu^{-1}(t_J)$ and $H(\sigma(\mathcal{D}), L) < \eta$, then there exists $\mathcal{B} \in C(\mu^{-1}(t_J))$ such that $L \in \mathcal{B}$ and $H^2(X(\mathcal{A} \cup \mathcal{D}, t_J), \mathcal{B}) < \frac{\varepsilon}{12}$.

Let $M \geq 1$ be such that $H^2(\mathcal{A}_M, \mathcal{A}) < \frac{\varepsilon}{12}$ and $H^2(\mathcal{D}, \mathcal{D}_M) < \eta$. Note that $H(\sigma(\mathcal{D}), \sigma(\mathcal{D}_M)) < \eta$. To prove this part, we take a point $x \in \sigma(\mathcal{D})$. By definition there exists $D \in \mathcal{D}$ such that $x \in D$, since $H^2(\mathcal{D}, \mathcal{D}_M) < \eta$, there is $D_M \in \mathcal{D}_M$ such that $H(D, D_M) < \eta$. So, there exists $d_M \in D_M \subset \sigma(\mathcal{D}_M)$ such that $d(x, d) < \eta$. Therefore, $x \in N(\eta, \sigma(\mathcal{D}_M))$. Thus, $\sigma(\mathcal{D}) \subset N(\eta, \sigma(\mathcal{D}_M))$. Similarly we can prove that $\sigma(\mathcal{D}_M) \subset N(\eta, \sigma(\mathcal{D}))$. Then $H(\sigma(\mathcal{D}), \sigma(\mathcal{D}_M)) < \eta$. Let $\mathcal{B} \in C(\mu^{-1}(t_J))$ be such that $\sigma(\mathcal{D}_M) \in \mathcal{B}$ and $H^2(X(\mathcal{A} \cup \mathcal{D}, t_J), \mathcal{B}) < \frac{\varepsilon}{12}$.

Let $\mathcal{G} = \bigcup\{C(G, t_0) : G \in \mathcal{B}\}$. By Lemma 3.3, $\mathcal{G} \in C(\mu^{-1}(t_0))$. Since $\sigma(\mathcal{D}_M) \in \mathcal{B}$ and $Q_M \in C(\sigma(\mathcal{D}_M), t_0)$, $Q_M \in \mathcal{G}$.

Now we prove that $H^2(\mathcal{A} \cup \mathcal{D}, \mathcal{G}) < \frac{\varepsilon}{4}$. Let $R \in \mathcal{A} \cup \mathcal{D}$. Since $\mu(\sigma(\mathcal{A} \cup \mathcal{D})) \geq t_J$, by Lemma 3.7, there exists $\mathcal{L} \in C(\mathcal{A} \cup \mathcal{D})$ such that $R \in \mathcal{L}$ and $\mu(\sigma(\mathcal{L})) = t_J$. Notice that $R \in C(\sigma(\mathcal{L}), t_0)$ (see [1: Lemma 3.1, p. 241]). So, $\mu(R) = t_0$, Thus, $\mu(\sigma(\mathcal{L})) - \mu(R) = t_J - t_0 < \delta$ and by the choice of δ, $H(\sigma(\mathcal{L}), R) < \frac{\varepsilon}{12}$. Since $\sigma(\mathcal{L}) \in X(\mathcal{A} \cup \mathcal{D}, t_J)$ and $H^2(X(\mathcal{A} \cup \mathcal{D}, t_J), \mathcal{B}) < \frac{\varepsilon}{12}$, there exists $F' \in \mathcal{B}$ such that $H(\sigma(\mathcal{L}), F') < \frac{\varepsilon}{12}$. Let $S \in C(F', t_0)$. Since $F' \in \mathcal{B}$, $S \in \mathcal{G}$. Since $\mathcal{B} \in C(\mu^{-1}(t_J))$ and $\mu(F') - \mu(S) < \delta$, by the choice of δ, $H(S, F') < \frac{\varepsilon}{12}$. Thus, $H(R, S) < \frac{\varepsilon}{4}$. Hence $R \in N(\frac{\varepsilon}{4}, \mathcal{G})$. On the other hand, let $G \in \mathcal{B}$ and $D \in C(G, t_0)$. Since $\mu(G) - \mu(D) < \delta$, by the choice of δ, $H(G, D) < \frac{\varepsilon}{12}$. Since $\mathcal{B} \subset N(\frac{\varepsilon}{12}, X(\mathcal{A} \cup \mathcal{D}, t_J))$, there exists $F_1 \in X(\mathcal{A} \cup \mathcal{D}, t_J)$ such that $H(G, F_1) < \frac{\varepsilon}{12}$. Since $F_1 \in X(\mathcal{A} \cup \mathcal{D}, t_J)$, there exists $\mathcal{L} \in C(\mathcal{A} \cup \mathcal{D})$ such that $F_1 = \sigma(\mathcal{L})$ and $\mu(\sigma(\mathcal{L})) = t_J$. Let $E_1 \in \mathcal{L}$. By [1: Lemma 3.1, p. 241], $E_1 \in C(F, t_0)$. Since $\mu(F_1) - \mu(E_1) = t_J - t_0 < \delta$, by the choice of δ, $H(E_1, F_1) < \frac{\varepsilon}{12}$. So, $H(D, E_1) < \frac{\varepsilon}{4}$. Thus, $D \in N(\frac{\varepsilon}{4}, \mathcal{A} \cup \mathcal{D})$. Hence $H^2(\mathcal{A} \cup \mathcal{D}, \mathcal{G}) < \frac{\varepsilon}{4}$.

Therefore, $H^2(\mathcal{A}_M, \mathcal{G}) \leq H^2(\mathcal{A}_M, \mathcal{A}) + H^2(\mathcal{A}, \mathcal{A} \cup \mathcal{D}) + H^2(\mathcal{A} \cup \mathcal{D}, \mathcal{G}) < \frac{\varepsilon}{2}$, a contradiction. □

Theorem 4.4. *Unicoherence is a sequential decreasing strong size property.*

Proof. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in [0, 1)$. Suppose that $\{t_j\}_{j \in \mathbb{N}} \subset (t_0, 1]$ is a sequence which converges to t_0 and each fiber $\mu^{-1}(t_j)$ is unicoherent.

Notice that $F_n(X)$ is unicoherent for each $n \geq 3$ (see [5: Theorem 8, p. 177]). So, since $\mu^{-1}(0) = F_n(X)$, $\mu^{-1}(0)$ is unicoherent for each $n \geq 3$.

In order to prove the other cases, we assume that $\mu^{-1}(t_0)$ is not unicoherent. Let $\mathcal{A}_1, \mathcal{A}_2 \in C(\mu^{-1}(t_0))$ be such that $\mu^{-1}(t_0) = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2$ is not connected. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two nonempty disjoint closed subsets of $\mu^{-1}(t_0)$ such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{F}_1 \cup \mathcal{F}_2$. Let $\varepsilon > 0$ be such that $N(\varepsilon, \mathcal{F}_1) \cap N(\varepsilon, \mathcal{F}_2) = \emptyset$.

For each $i \in \{1, 2\}$, let $\mathcal{B}_i = \mathcal{A}_i \setminus (N(\varepsilon, \mathcal{F}_1) \cup N(\varepsilon, \mathcal{F}_2))$. Notice that $\mathcal{B}_1$ and $\mathcal{B}_2$ are nonempty disjoint closed subsets of $\mu^{-1}(t_0)$. Let $0 < \varepsilon_1 < \frac{\varepsilon}{8}$ be such that $N(\varepsilon_1, \mathcal{B}_1) \cap N(\varepsilon_1, \mathcal{B}_2) = \emptyset$. Let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon_1}{2}$. Let $t_J \in (t_0, t_0 + \delta)$. For each $i \in \{1, 2\}$, let $\mathcal{C}_i = \bigcup\{C_D^{t_J} : D \in \mathcal{A}_i\}$.

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We prove that $\mu^{-1}(t_J) = \mathcal{C}_1 \cup \mathcal{C}_2$. Let $P \in \mu^{-1}(t_J)$. Using order arcs, it can be shown that there exists $Q \in \mu^{-1}(t_0)$ such that $P \in C_Q^{t_J}$. So, $P \in \mathcal{C}_1 \cup \mathcal{C}_2$. On the other hand, by Lemma 3.3, $\mathcal{C}_1, \mathcal{C}_2 \in C(\mu^{-1}(t_J))$.

For each $i \in \{1,2\}$, let

$$\mathcal{G}_i = \{F \in \mu^{-1}(t_J) : \text{ there exists } A \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_i)) \text{ such that } F \in C_A^{t_J}\}.$$

We will show that $\mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{G}_1 \cup \mathcal{G}_2$. Let $D \in \mathcal{C}_1 \cap \mathcal{C}_2$. For each $i \in \{1,2\}$, let $A_i \in \mathcal{A}_i$ be such that $D \in C_{A_i}^{t_J}$. By the choice of δ, $H(A_1, A_2) \le H(A_1, D) + H(A_2, D) < \varepsilon_1$. By the choice of ε_1, $\{A_1, A_2\} \cap (N(\frac{\varepsilon}{8}, \mathcal{F}_1) \cup N(\frac{\varepsilon}{8}, \mathcal{F}_2)) \neq \emptyset$. We assume, without loss of generality, that $A_1 \in N(\frac{\varepsilon}{8}, \mathcal{F}_1) \cup N(\frac{\varepsilon}{8}, \mathcal{F}_2)$. So, $D \in \mathcal{G}_1 \cup \mathcal{G}_2$.

In will prove that $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$. Let $G \in \mathcal{G}_1 \cap \mathcal{G}_2$. Then there exist $E_1 \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_1))$ and $E_2 \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_2))$ such that $G \in C_{E_1}^{t_J} \cap C_{E_2}^{t_J}$. By the choice of δ, $H(E_1, E_2) \le H(E_1, G) + H(E_2, G) < \varepsilon_1$. For $i \in \{1,2\}$, let $F_i \in \mathcal{F}_i$ be such that $H(E_i, F_i) < \frac{\varepsilon}{4}$. Thus, $H(F_1, F_2) < \varepsilon$ which contradicts the choice of ε. We have shown that $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$.

Note that, given $F_i \in \mathcal{F}_i$, there exists $D_i \in \mu^{-1}(t_J)$ such that $D_i \in C_{F_i}^{t_J}$. Thus, $D_i \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_i$. We have shown that $\mathcal{G}_1$ and $\mathcal{G}_2$ are disjoint subsets of $\mu^{-1}(t_J)$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_1 \neq \emptyset \neq \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_2$.

Now, we prove that $\mathcal{G}_1$ is closed. Let $\{B_k\}_{k\in\mathbb{N}}$ be a sequence of $\mathcal{G}_1$ such that $\lim B_k = B$ for some $B \in \mu^{-1}(t_J)$. Notice that for each $k \in \mathbb{N}$, there exists $A_k \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_1))$ such that $B_k \in C_{A_k}^{t_J}$. By compactness we may assume that $\lim A_k = A$ for some $A \in \mathrm{cl}(N(\frac{\varepsilon}{8}, \mathcal{F}_1))$. Since $B_k \in C_{A_k}^{t_J}$ for each $k \in \mathbb{N}$, $A \subset B$. By Lemma 3.1, $B \in C_A^{t_J}$. Hence $B \in \mathcal{G}_1$. Thus, $\mathcal{G}_1$ is closed. Similarly we can prove that $\mathcal{G}_2$ is closed.

Then $\mathcal{C}_1 \cap \mathcal{C}_2$ is disconnected. Therefore, $\mu^{-1}(t_J)$ is not unicoherent, a contradiction. □

It is known that for every $n > 1$, $F_n(X)$ is aposyndetic for every continuum X, and we know that every aposyndetic continuum is decomposable (see [4: Theorem 4, p. 289]). Thus, if X is a continuum and μ is a strong size map defined on $C_n(X)$, then $\mu^{-1}(0) = F_n(X)$. Hence $\mu^{-1}(0)$ is decomposable. Therefore, indecomposability is not a sequential decreasing strong size property.

Theorem 4.5. *Indecomposability is an almost sequential decreasing strong size property.*

Proof. Let μ be a strong size map for $C_n(X)$ and let $t_0 \in (0,1)$. Suppose that $\{t_j\}_{j\in\mathbb{N}} \subset (t_0, 1]$ is a sequence which converges to t_0 and each fiber $\mu^{-1}(t_j)$ is indecomposable.

Suppose that there are two proper subcontinua $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mu^{-1}(t_0)$ such that $\mu^{-1}(t_0) = \mathcal{A}_1 \cup \mathcal{A}_2$. Let $A_1 \in \mathcal{A}_1 \setminus \mathcal{A}_2$ and $A_2 \in \mathcal{A}_2 \setminus \mathcal{A}_1$. Let $\varepsilon > 0$ be such that $\mathcal{V}_\varepsilon^H(A_1) \cap \mathcal{A}_2 = \emptyset = \mathcal{V}_\varepsilon^H(A_2) \cap \mathcal{A}_1$. Let $\delta > 0$ be as in Lemma 3.2 for the number $\frac{\varepsilon}{2}$. Take $t_J \in (t_0, t_0 + \delta)$. For each $i \in \{1,2\}$, put $\mathcal{G}_i = \bigcup\{C_A^{t_J} : A \in \mathcal{A}_i\}$. We show that $\mu^{-1}(t_J) = \mathcal{G}_1 \cup \mathcal{G}_2$. Let $E \in \mu^{-1}(t_J)$. Using order arcs, it can be shown that there exists $F \in \mu^{-1}(t_0)$ such that $E \in C_F^{t_J}$. So, $E \in \mathcal{G}_1 \cup \mathcal{G}_2$. On the other hand, by Lemma 3.5, $\mathcal{G}_1, \mathcal{G}_2 \in C(\mu^{-1}(t_J))$.

Fix $G \in C_{A_1}^{t_J}$. If $G \in \mathcal{G}_2$, then $G \in C_R^{t_J}$ for some $R \in \mathcal{A}_2$. Since $\mu(G) - \mu(A_1)$, $\mu(G) - \mu(R) < \delta$, by the choice of δ, $H(R, G) < \frac{\varepsilon}{2}$ and $H(G, A_1) < \frac{\varepsilon}{2}$. So, $H(A_1, R) < \varepsilon$ which contradicts the choice of ε. Hence $\mathcal{G}_2 \neq \mu^{-1}(t_J)$. Similarly, $\mathcal{G}_1 \neq \mu^{-1}(t_J)$. Thus, $\mu^{-1}(t_J)$ is descomposable, a contradiction.
Therefore, $\mu^{-1}(t_0)$ is indecomposable. □

Acknowledgement. The authors thank the referee for the suggestions made that improve the paper.

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Received 14. 10. 2016
Accepted 1. 9. 2017

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#4372 SUMMARY

SUMMARY REVIEW EDITING

SUBMISSION

Authors	FELIX CAPULIN PEREZ, ALEJANDRO FUENTES MONTES DE OCA, MIGUEL ANGEL LARA MEJIA, FERNANDO OROZCO ZITLI
Title	INCREASING STRONG SIZE PROPERTIES AND BLOCK STRONG SIZE PROPERTIES
Original file	4372-13547-1-SM.PDF 2019-05-03
Supp. files	None ADD A SUPPLEMENTARY FILE
Submitter	FELIX CAPULIN PEREZ
Date submitted	May 3, 2019 - 12:15 AM
Section	General Topology
Editor	None assigned

STATUS

Status	Awaiting assignment
Initiated	2019-05-03
Last modified	2019-05-03

SUBMISSION METADATA

EDIT METADATA

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TITLE AND ABSTRACT

Title INCREASING STRONG SIZE PROPERTIES AND BLOCK STRONG SIZE PROPERTIES

Abstract Let X be a continuum. The n-fold hyperspace $C_{n}(X)$, $n<\infty$, is the family of all nonempty closed subsets of X with at most n components, topologized with the Hausdorff metric. A topological property $\mathcal{P}$ is said to be an increasing strong size property provided that if μ is a strong size map for $C_{n}(X)$ and $t_{0}\in [0,1)$ is such that $\mui(t_{0})$ has property $\mathcal{P}$, then so does $\mui(t)$ for each $t\in(t_{0},1)$. A \textit{strong size block} is the subset $\mui([s,r])$ for a strong size map μ and $0\leq s< r\leq 1$. In this paper we show that uniform pathwise connectedness, uniform continuum-chainability and local connectedness are increasing strong size properties and we will show some block strong size properties.

INDEXING

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INCREASING STRONG SIZE PROPERTIES AND BLOCK STRONG SIZE PROPERTIES

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Abstract. Let X be a continuum. The n-fold hyperspace $C_n(X)$, $n < \infty$, is the family of all nonempty closed subsets of X with at most n components, topologized with the Hausdorff metric. A topological property $\mathcal{P}$ is said to be a increasing strong size property provided that if μ is a strong size map for $C_n(X)$ and $t_0 \in [0, 1)$ is such that $\mu^{-1}(t_0)$ has property $\mathcal{P}$, then so does $\mu^{-1}(t)$ for each $t \in (t_0, 1)$. A *strong size block* is the subset $\mu^{-1}([s, r])$ for a strong size map μ and $0 \leq s < r \leq 1$. In this paper we show that uniform pathwise connectedness and uniform continuum-chainability and local connectedness are increasing strong size properties and we will show some block strong size properties.

1. Introduction and preliminaries

A *continuum* is a nonempty compact, connected, metric space. A *subcontinuum* of a space Z is a continuum contained in Z. The symbol $\mathbb{N}$ denotes the set of positive integers. Let X be a continuum. For each $n \in \mathbb{N}$, $C_n(X)$ denotes the hyperspace of all nonempty closed subsets of X with at most n components; it is called the *n-fold hyperspace* of X, note that $C_1(X)$ is the classical hyperspace of all subcontinua of X and, as is customary, it is denoted by $C(X)$ instead of $C_1(X)$. The set $F_n(X) = \{A \in C_n(X) : A \text{ has at most } n \text{ points}\}$ is called the *n-fold symmetric product* of a continuum X. We topologize these sets with the Hausdorff metric H, defined as follows: $H(A, B) = \inf\{\varepsilon > 0 : A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\}$ (see [13, p. 1]). A *map* means a continuous function. A *Whitney map* for $C_n(X)$ is a map $\sigma : C_n(X) \to [0, 1]$ such that $\sigma(\{x\}) = 0$ for each $x \in X$ and if $A \subset B$ and $B \neq A$, then $\sigma(A) < \sigma(B)$. A *strong size map* for $C_n(X)$ is a map $\mu : C_n(X) \to [0, 1]$ such that $\mu(A) = 0$ for each $A \in F_n(X)$ and if $A \subset B$, $A \neq B$ and $B \notin F_n(X)$, then $\mu(A) < \mu(B)$ H. Hosokawa introduced in [4]) strong size maps on the n-fold hyperspace of a continuum as a generalization of Whitney maps for the hyperspace of subcontinua of a continuum. Throughout of this paper the symbol μ will represent a strong size map for $C_n(X)$ and the symbol σ will represent a Whitney map for $C(X)$. A *strong size level (Whitney level)* is the subset $\mu^{-1}(t)$ ($\sigma^{-1}(t)$) for $t \in [0, 1]$. A *strong size block* is the subset $\mu^{-1}([s, r])$, where $0 \leq s < r \leq 1$. Strong size levels and strong size blocks are always continua (see [4, Theorem 2.10, p.958]).

Let $\mathcal{P}$ be topological property. The property $\mathcal{P}$ is called:

- *Strong size property* if whenever X has property $\mathcal{P}$, so does every strong size level.
- *Increasing Whitney property* provided that if $t_0 \in [0, 1)$ and $\sigma^{-1}(t_0)$ has property $\mathcal{P}$, then $\sigma^{-1}(t)$ has property $\mathcal{P}$ for each $t \in [t_0, 1)$.
- *Increasing strong size property* provided that if $t_0 \in [0, 1)$ and $\mu^{-1}(t_0)$ has property $\mathcal{P}$, then $\mu^{-1}(t)$ has property $\mathcal{P}$ for each $t \in (t_0, 1)$.

2010 *Mathematics Subject Classification.* Primary, 54C05, 54C10, 54B20; Secondary, 54B15.
Key words and phrases. Continuum, n-fold hyperspace, strong size map, increasig strong size property, block strong size map.

- *Block strong size property* if whenever X has property $\mathcal{P}$, so does every every strong size block.
- *Almost block strong size property* if whenever X has property $\mathcal{P}$, so does every every strong size block $\mu^{-1}([s,r])$, where $0 < s < r \leq 1$.

The interested reader on increasing Whitney properties are referred to [5, 6, 9, 14, 15]. Note that increasing strong size properties are the natural generalization of increasing Whitney properties. In this paper we prove that pathwise connectedness, continuum-chainability and being a locally connected continuum are increasing strong size properties, and we also show that being an arc, begin a simple closed curve, being the Buckethandle continuum, having span zero, chainability, being a particular solenoid, being a particular pseudo-solenoid, being proper circle-like, being the pseudo-arc and being the pseudo-circle are not strong size properties. Additionally we show some block strong size properties for a continuum.

Let X be a continuum. If $A \in C_n(X)$, let $\mathcal{O}(A) = \{B \in C_n(X) : \text{each component of } B \text{ intersects } A\}$ and let $\mathcal{O}^*(A) = \{B \in \mathcal{O}(A) : A \subset B\}$. If $t \in [0,1)$ and $\mu(A) > t$, let $C(A,t) = \{B \in \mu^{-1}(t) : B \in \mathcal{O}^*\}$.

The following two lemma appear in [2].

Lemma 1.1. [2, Lemma 3.1, p. 1143] *Let X be a continuum and let $\{A_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ be sequences of $C_n(X)$ such that $A_k \to A$, $B_k \to B$. If for each $k \in \mathbb{N}$, $B_k \in \mathcal{O}(A_k)$, then $B \in \mathcal{O}(A)$.*

Lemma 1.2. [2, Lemma 3.2, p. 1143] *Let X be a continuum. Whether each $\varepsilon > 0$, there exists $\delta > 0$ such that if $A, B \in C_n(X)$ satisfy that $B \in \mathcal{O}(A)$, $A \subset N(\delta, B)$ and $|\mu(A) - \mu(B)| < \delta$, then $H(A,B) < \varepsilon$.*

Let $X_1, X_2, \ldots, X_m$ be a finite collection of subsets of X. We define the subset $< X_1, X_2, \ldots, X_m >_n$ of $C_n(X)$ by $< X_1, X_2, \ldots, X_m >_n = \{A \in C_n(X) : A \subset X_1 \cup \cdots \cup X_n \text{ and } A \cap X_k \neq \emptyset \text{ for } k \in \{1, \ldots, n\}\}$. Note that if $Y_1, Y_2, \ldots, Y_m$ are subset of X and $X_i \subset Y_i$ for each $i \in \{1, 2, \ldots, m\}$, then $< X_1, X_2, \ldots, X_m >_n \subset < Y_1, Y_2, \ldots, Y_m >_n$. If $X_1, X_2, \ldots, X_m$ are closed subsets of X, then $< X_1, X_2, \ldots, X_m >_n$ is closed in $C_n(X)$. It is known that the collection of all $< U_1, U_2, \ldots, U_m >_n$, where $U_1, U_2, \cdots, U_m$ are open subsets of X, is a base for the topology of $C_n(X)$ (see [10]).

2. Pathwise connectedness and continuum-chainability

A continuum X is said to be *uniformly pathwise connected* if there exists a family $\mathcal{F}$ of paths in X, fulfilling:

(1) For each pair of points $x, y \in X$ there exists a path $\alpha \in \mathcal{F}$ such that $\alpha(0) = x$ and $\alpha(1) = y$,
(2) for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for each $\alpha \in \mathcal{F}$ there are numbers $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\text{diam}(\alpha([t_{i-1}, t_i])) < \varepsilon$ for each $i \in \{0, 1, \ldots, k\}$.

A finite family of sets $\{A_1, \ldots A_m\}$ is said to be a *weak chain* provided that $A_i \cap A_j \neq \emptyset$ if $|i-j| \leq 1$. A continuum X is said to be *continuum-chainable* provided that for each $\varepsilon > 0$ and for each pair of points $p, q \in X$, there exists a weak chain of continua $\{A_1, \ldots, A_m\}$ such that $p \in A_1$, $q \in A_m$ and $\text{diam}(A_i) < \varepsilon$ for each $i \in \{1, \ldots, m\}$, .

A continuum X is said to be *uniformly continuum-chainable* provided that for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for each pair of points $p, q \in X$, there exists a weak chain of continua $\{A_1, \ldots A_k\}$ such that $p \in A_1$, $q \in A_k$ and $\text{diam}(A_i) < \varepsilon$ for each $i \in \{1, \ldots, k\}$.
In this section we will prove that pathwise connectedness, continuum-chainability are increasing strong size properties. We start with the following result

The proof of the next theorem is based on the idea in the proof of [8, Lemma 2, p. 2].

Lemma 2.1. *Let $t \in (0,1)$. Then for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ satisfying that if $A, B \in \mathcal{O}^*(D) \cap \mu^{-1}(t)$ for some $D \in F_n(X)$, then there exists a path $q : [0,1] \to \mu^{-1}(t)$ such that $q(0) = A$, $q(1) = B$ and $\operatorname{diam}(q([\frac{i-1}{m}, \frac{i}{m}])) < \varepsilon$ for each $i \in \{0, 1, \ldots, m\}$.*

Proof. Let $D \in F_n(X)$ and let $A, B \in \mathcal{O}^*(D) \cap \mu^{-1}(t)$. First of all, we will prove that always there exist a path from A to B. Let $\alpha', \beta' : [0,1] \to C_n(X)$ be two order arcs such that $\alpha'(0) = \beta'(0) = D$, $\alpha'(1) = A$ and $\beta'(1) = B$. Let α and β be parametrizations of these arcs by using the strong size map, i.e., $\alpha : [0,t] \to \alpha'([0,1])$ and $\beta : [0,t] \to \beta'([0,1])$, such that $\mu(\alpha(u)) = \mu(\beta(u)) = u$ for each $u \in [0,t]$. Given $s \in [0,t]$. Define $f_s : [0,t] \to C_n(X)$ by $f_s(r) = \alpha(s) \cup \beta(r)$. Then f_s is well defined and it is continuous. Since $\mu(f_s(0)) \leq t$ and $\mu(f_s(1)) \geq t$, there exists $r \in [0,t]$ such that $\mu(f_s(r)) = t$. Set $r_s = \sup\{r \in [0,t] : \mu(f_s(r)) = t\}$. So, we define $\gamma : [0,t] \to \mu^{-1}(t)$ by $\gamma(s) = \alpha(s) \cup \beta(r_s)$. It is easy to prove that γ is well defined and continuous. Hence, γ is a path in $\mu^{-1}(t)$ joining A and B.
Now, let $\varepsilon > 0$. Let $\delta > 0$ be as in the Lemma 1.2 for the number ε. Let $\delta' > 0$ be as in the Lemma 1.2 for the number δ. Let $k \in \mathbb{N}$ such that $\frac{t}{k} < \delta'$. Take $i \in \{0, 1, \ldots, k\}$. In order to prove that $\operatorname{diam}(\gamma([\frac{t(i-1)}{k}, \frac{it}{k}])) < \varepsilon$, let $v, w \in [\frac{t(i-1)}{k}, \frac{it}{k}]$ with $v < w$. Then $\alpha(v) \subset \alpha(w)$ and $\mu(\alpha(w)) - \mu(\alpha(v)) = w - v < \delta'$. By the choice of δ', $H(\alpha(v), \alpha(w)) < \delta$.
We need to prove that $r_w \leq r_v$. Suppose there exists $l \in \{r \in [0,t] : \mu(f_w(r)) = t\}$ such that $r_v < l$. Since $\beta(r_v) \subset \beta(l)$ and $\alpha(v) \subset \alpha(w)$, $f_v(r_v) \subset f_v(l) \subset f_w(l)$. Using $\mu(f_v(r_v)) = \mu(f_w(l)) = t$, we have $f_v(r_v) = f_v(l)$ and $\mu(f_v(l)) = t$, a contradiction. Thus $r_w \leq r_v$.
Since $\beta(r_w) \subset \beta(r_v)$ and $\alpha(w) \subset N(\delta, \alpha(v))$, $f_w(r_w) \subset N(\delta, f_v(r_v))$, which means that $\gamma(w) \subset N(\delta, \gamma(v))$. By the choice of δ, $H(\gamma(v), \gamma(w)) < \varepsilon$.
Therefore, the map $q : [0,1] \to \mu^{-1}(t)$ defined by $q(l) = \gamma(lt)$ has the required properties. □

Recall the following. The function $u : C(C_n(X)) \to C_n(X)$ given by $u(\mathcal{A}) = \bigcup\{A : A \in \mathcal{A}\}$ is a map called the union map.
The following theorem is similar to a result for Whitney levels (see [14, Theorem 3.1]).

Theorem 2.2. *Let X be a continuum and let $t_0 \in [0,1)$. If $\mu^{-1}(t_0)$ is uniformly continuum-chainable, then $\mu^{-1}(t)$ is uniformly pathwise connected for each $t \in (t_0, 1)$.*

Proof. Let $t \in (t_0, 1)$. We will prove that $\mu^{-1}(t)$ is uniformly pathwise connected. Since the composition of μ and the union map u is uniformly continuous, there exists $\delta > 0$ such that if $\mathcal{A} \in C(\mu^{-1}(t_0))$ and $\operatorname{diam}(\mathcal{A}) < \delta$ then $\mu(u(\mathcal{A})) < t$. Let k be as in the definition of the uniform continuum-chainability of $\mu^{-1}(t_0)$ for the number δ.
Let $A, B \in \mu^{-1}(t)$. Using order arcs, we find A^* y $B^* \in \mu^{-1}(t_0)$ such that $A \in \mathcal{O}^*(A^*)$ and $B \in \mathcal{O}^*(B^*)$. Then there exists $\{\mathcal{A}_1, \ldots, \mathcal{A}_k\}$ be a weak chain of subcontinua of $\mu^{-1}(t_0)$ such that $A^* \in \mathcal{A}_1$ and $B^* \in \mathcal{A}_k$ and $\operatorname{diam}(\mathcal{A}_i) < \delta$ for each $i \in \{1, \ldots, k\}$. Notice that $\mu(u(\mathcal{A}_i)) < t$ for each $i \in \{1, \ldots, k\}$. Using orders arcs, we can find an element $A_i \in \mu^{-1}(t)$ such that $u(\mathcal{A}_i) \subset A_i$. Clearly $\{A, A_1, \ldots, A_k, B\}$ is a weak chain.
Applying the Lemma 2.1 to each one of the pairs A and A_1, A_1 and A_2, …, and A_k and B, we find a path $\alpha : [0,1] \to \mu^{-1}(t)$ between A and B and a sequence of numbers $0 = r_0 < r_1 \cdots < r_{km} = 1$ such that $\operatorname{diam}(\alpha(r_{i-1}, r_i)) < \epsilon$. □

Note that the proof of [8, Theorem 2, p. 170]), shows that a continuum uniformly pathwise connected is a continuum uniformly continuum-chainable. From this fact and Theorem 2.2, the following result is immediate.

Theorem 2.3. *Uniform pathwise connectedness and uniform continuum-chainability are increasing strong size properties.*

The metod used in the proof of Theorem 2.2 can be used to prove the following theorem.

Theorem 2.4. *Let X be a continuum and let $t_0 \in [0,1)$. If $\mu^{-1}(t_0)$ is chainable-continuum, then $\mu^{-1}(t)$ is pathwise connected for each $t \in (t_0, 1)$.*

Since pathwise connectedness implies continuum-chainability, the following corollary is a consequence of Theorem 2.4.

Corollary 2.5. *Pathwise connectedness and continuum-chainability are increasing strong size properties.*

It is known that pathwise connectedness is equivalent to arcwise connectedness in continua. Thus we have the following result.

Corollary 2.6. *Arcwise connectedness is an increasing strong size property.*

Corollary 2.7. *Arcwise connectedness is a strong size property.*

3. Block strong size properties

The main of this section is to prove that the following properties are block strong size properties: arcwise connectedness, locally connectedness, aposyndesis. And the property of being countable closed set aposyndetic is an almost block strong size property.

Recall that a map $g : Z \to S^1$, from a topological connected space into the unit circle $S^1 \subset \mathbb{R}^2$, has a *lifting* if there exists a map $h : Z \to \mathbb{R}$ such that $f = exp \circ h$. A connected topological space Z has *property* (b) if each map $f : Z \to S^1$ has a lifting.

Let $A \in C_n(X)$, $\mathcal{K}_A$ will denote the set $\{B \in C_n(X) : A \subset B\}$. The statement in [1, Lemma 13, p. 2004] was shown for $C(X)$ and 2^X, we affirm that it is valid for $C_n(X)$. The proof of the next result is based on the idea in the proof of [1, Lemma 13, p. 2004].

Lemma 3.1. *Let $\mathcal{K}$ be a nonempty subset of $C_n(X)$. If $\mathcal{K}_A \subset \mathcal{K}$, for each $A \in \mathcal{K}$, then $\mathcal{K}$ has property* (b).

Proof. First, it is easy to see that if α is an order arc in 2^X and $\alpha(0) \in C_n(X)$, then $\alpha(t) \in C_n(X)$ for all $t \in [0,1]$. Thus, this guarantees the existence or order arcs in $C_n(X)$. Claim: let $A \in C_n(X)$ and let $g : \mathcal{K}_A \to S^1$ be a map. Suppose that α and β are order arcs from A to X in $C_n(X)$. If h_α and h_β are liftings of $g|_\alpha$ and $g|_\beta$, respectively, such that $h_\alpha(X) = h_\beta(X)$, then $h_\alpha(A) = h_\beta(A)$. the proof of this fact is the same as [1, Lemma 12, p. 2004].

Finally, let $f : \mathcal{K} \to S^1$ be a map, we show that f has a lifting. Fix a number $t_0 \in exp^{-1}(f(X))$. Let $A \in \mathcal{K}$ and let α_A be an order arc from A to X in $C_n(X)$. Since α_A is homeomorphic to $[0,1]$ (see [13, Lemma 1.3, p. 57], we have that α_A has property (b). Let $h_A : \alpha_A \to \mathbb{R}$ the unique map such that $f|_{\alpha_A} = exp \circ h_A$ and $h_A(X) = t_0$. We define $h : \mathcal{K} \to \mathbb{R}$ by $h(A) = h_A(A)$. By the claim, h is well defined and, clearly, $f = exp \circ h$. The continuity of h is the same as the case $C(X)$ and 2^X, as we had warned. Therefore, $\mathcal{K}$ has property (b). □

Theorem 3.2. *Let X be a continuum. Let $t \in [0,1]$. The block $\mu^{-1}([t,1])$ has property* (b).

Proof. Notice that for each $A \in \mu^{-1}([t,1])$, we have that $\{B \in \mu^{-1}([t,1]) : A \subset B\} \subset \mu^{-1}([t,1])$. Thus, by Lemma 3.1, $\mu^{-1}([t,1])$ has property (b). □

Recall that a connected topological space Z is *unicoherent* provided that if $Z = A \cup B$, where A and B are closed connected subsets of Z, then $A \cap B$ is connected. We say that Z is hereditarily unicoherent provided that every subcontinuo of Z is unicoherent.

Since property (b) implies unicoherence (see [16, Theorem 7.3, p. 227]), the following corollary is a consequence of Theorem 3.2.

Corollary 3.3. *Let X be a continuum. Let $t \in [0,1]$. The hyperspace $\mu^{-1}([t,1])$ is unicoherent.*

Theorem 3.4. *Let X be a continuum. Let $0 \leq t < s \leq 1$. If there exists $t_0 \in [t,s]$ such that $\mu^{-1}(t_0)$ is arcwise connected, then $\mu^{-1}([t,s])$ is arcwise connected.*

Proof. In order to prove the result, it is enough to give an order arc from an element of $\mu^{-1}([t,s])$ to $\mu^{-1}(t_0)$. Let $A \in \mu^{-1}([t,s])$ such that $A = A_1 \cup \cdots \cup A_n$ where $A_1, A_2, \ldots, A_n$ are the components of A. Let $x_i \in A$ for $i = 1, ..., n$, then there exists an order arc α in $C_n(X)$ from $\{x_1, ..., x_n\}$ to X. Since $\mu \circ \alpha$ is a map, there exists $m_r \in [0,1]$ such that $\mu(\alpha(m_r)) = r$ for each $r \in [0,1]$. If $\mu(A) = r_A$, then $\alpha(m_{r_A}) = A$. We observe that $\alpha(m_{t_0}) \in \mu^{-1}(t_0)$. If $m_{r_A} \leq m_{t_0}$, then $\alpha|_{[m_{r_A}, m_{t_0}]}$ is an order arc from A to $\mu^{-1}(t_0)$. If $m_{t_0} \leq m_{r_A}$, then $\alpha|_{[m_{t_0}, m_{r_A}]}$ is an order arc from $\mu^{-1}(t_0)$ to A. □

It is known by [4, Theorem 3.3, p. 963] that the property of being an arcwise connected continuum is a strong size property. So we have the following collorary.

Corollary 3.5. *Let X be a continuum arcwise connected, $n \in \mathbb{N}$ and let $\mu : C_n(X) \to [0,1]$ be a strong size map for $C_n(X)$. If $0 \leq t < s \leq 1$, then $\mu^{-1}([t,s])$ is arcwise connected.*

A continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua. We say that Z is hereditarily decomposable provided that every subcontinuum nondegenerate of Z is decomposable.

Lemma 3.6. *Let X be a hereditarily decomposable and hereditarily unicoherent continuum and let $t \in (0,1)$. Then $\mu^{-1}(t)$ contains a $(2n-1)$-cell.*

Proof. Let $A \in \mu^{-1}(t)$ such that $A = A_1 \cup \cdots \cup A_n$ where $A_1, A_2, \ldots, A_n$ are the components of A. For each $i \in \{1, 2, \ldots, n\}$, A_i is a descomposable and unicoherent subcontinuum of X. Suppose that $M_i = E_i \cup F_i$, where E_i and F_i are proper subcontinua of M_i, for each $i \in \{1, 2, \ldots, n\}$. For each $i \in \{1, 2, \ldots, n\}$, let $\alpha_i : [0,1] \to C(X)$ and $\beta_i : [0,1] \to C(X)$ be order arcs such that $\alpha_i(0) = \beta_i(0) = E_i \cap F_i$, $\alpha_i(\frac{1}{2}) = E_i$, $\beta_i(\frac{1}{2}) = F_i$ and $\alpha_i(1) = \beta_i(1) = X$. Define the function $\xi : [0,1]^{2n} \to C_n(X)$ by $\xi(t_1, \ldots, t_{2n}) = \bigcup_{i=1}^{n} \alpha_i(t_{2i-1}) \cup \beta_i(t_{2i})$. Then ξ is well defined and continuous.
The proof used in [11, Theorem 3.2, p. 470] can be used in proving that $\mu^{-1}(t)$ contains a $(2n-1)$-cell.

□

A continuum is said to be *irreducible* provided that there exist two points in X such that there is no a proper subcontinuum containing both points.

Proposition 3.7. *There exists a irreducible continuum such that its stronger size levels does not.*

Proof. Let I the arc $[0,1]$. It is known that $F_n(I)$ contains a n-cell. By Corollary 2.7, $\mu^{-1}(t)$ is an arcwise continuum. Since I is a hereditarily decomposable and hereditarily unicoherent continuum, by Lemma 3.6, $\mu^{-1}(t)$ contains a $(2n-1)$-cell for each $t \in (0,1)$. Thus $\mu^{-1}(t)$ is not irreducible for each $t \in [0,1]$. □

Corollary 3.8. *Irreducibility is not a strong size property.*

Corollary 3.9. *The following properties are not strong size properties for $n \geq 3$:*

(1) being an arc,
(2) begin a simple closed curve,
(3) being the Buckethandle continuum,
(4) having span zero,
(5) chainability,

(6) being a particular solenoid,
(7) being a particular pseudo-solenoid,
(8) being proper circle-like,
(9) being the pseudo-arc,
(10) being the pseudo-circle.

Proof. By [11, Theorem 3.2, p. 470] each strong size level contains a $(2n-1)$-cell. Since each of the continua described in (1)-(10) has the property that its nondegenerate proper subcontinua are hereditarily irreducible(see [13, p. 201] and [12, Theorem 12.5, p. 233]), the properties (1)-(10) are not a strong size properties. □

We do not if there are continua such that its strong size levels are irreducible. In this way we perform the following question.

Question 3.10. Does there exist a continuum with its strong size levels irreducible?

Lemma 3.11. *Let X be a continuum and let $0 \le t_0 < t$. If $\mathcal{K}$ is a subcontinuum of $C_n(X)$, $D \in \mu^{-1}(t)$ and $C(D,t_0) \subset \mathcal{K}$, then $D \in C(\sigma(\mathcal{K}),t)$.*

Proof. Notice that $\bigcup C(D,t_0) = D$. Thus $D \subset \sigma(\mathcal{K})$. Let $A \in C(D,t_0)$. Since $A \in \mathcal{K}$, by [3, Lemma 3.1] each component of $\sigma(\mathcal{K})$ intersects A. Let K_C be a component of $\sigma(\mathcal{K})$, since $A \subset D$, $K_C \cap D \neq \emptyset$. So, $D \in C(\sigma(\mathcal{K}),t)$. □

Lemma 3.12. *Let X be a continuum, and let $0 \le t_0 < s \le r \le t$. If $\mathcal{K}$ is an open subset of $\mu^{-1}(t_0)$, then $\mathcal{G} = \{D \in \mu^{-1}(r) : C(D,t_0) \subset \mathcal{K}\}$ is an open subset of $\mu^{-1}([s,t])$.*

Proof. We will prove that $\mu^{-1}([s,t]) \setminus \mathcal{G}$ is a closed subset of $\mu^{-1}([s,t])$. We can assume that $\mu^{-1}([s,t]) \neq \mathcal{G}$ and $\mathcal{G} \neq \emptyset$. Let $\{B_k\}_{k\in\mathbb{N}}$ be a sequence of $\mu^{-1}([s,t]) \setminus \mathcal{G}$ such that converge for some $B \in \mu^{-1}([s,t])$. Then, for each $k \in \mathbb{N}$, $C(B_k,t_0,n) \cap \mu^{-1}(t_0) \setminus \mathcal{K} \neq \emptyset$. By [4, Theorem 2.14], for each $k \in \mathbb{N}$, $C(B_k,t_0) \in C(\mu^{-1}(t_0))$.

By compactness we may assume that the sequence $\{C(B_k,t_0)\}_{k\in\mathbb{N}}$ converges to an element $\mathcal{D} \in C(\mu^{-1}(t_0))$. Since $C(B_k,t_0) \cap \mu^{-1}(t_0) \setminus \mathcal{K} \neq \emptyset$, for each k, and $\mu^{-1}(t_0) \setminus \mathcal{K}$ is closed, we conclude that $\mathcal{D} \cap \mu^{-1}(t_0) \setminus \mathcal{K} \neq \emptyset$. Now, let $D \in \mathcal{D}$. Since $\lim C(B_k,t_0) = \mathcal{D}$, there exists $D_k \in C(B_k,t_0)$ for each $k \in \mathbb{N}$ such that $\lim D_k = D$. It follows of Lemma 1.1 that $D \in C(B,t_0)$. Then $C(B,t_0) \cap \mu^{-1}(t_0) \setminus \mathcal{K} \neq \emptyset$ and $B \in \mu^{-1}([s,t]) \setminus \mathcal{G}$. Therefore $\mathcal{G}$ is an open subset of $\mu^{-1}([s,t])$. □

Theorem 3.13. *Let X be a continuum and let $0 \le t_0 \le s < t \le 1$. If $\mu^{-1}(t_0)$ is a locally connected continuum, then $\mu^{-1}([s,t])$ is a locally connected continuum.*

Proof. Let $A = A_1 \cup \cdots \cup A_s \in \mu^{-1}([s,t])$, where A_i, $i \in \{1,\ldots,s\}$ are the components of A. Let $\varepsilon > 0$ such that $\mathcal{V}(\varepsilon,A)$ is a neighborhood of A in $\mu^{-1}([s,t])$. Let $\delta > 0$ be as in the Lemma 1.2 for the number ε. Fix $0 < \gamma < \frac{\delta}{2}$ such that $\mathrm{cl}(N(\gamma,A_i)) \cap \mathrm{cl}(N(\gamma,A_j)) = \emptyset$ if $i \neq j$.
Let $\mathcal{U} = \{D \in \mu^{-1}(t_0) : D \subset N(\frac{\gamma}{2},A)\}$. Notice that $\mathcal{U}$ is an open subset of $\mu^{-1}(t_0)$ and $C(A,t_0,n) \subset \mathcal{U}$. Let $\mathcal{K}$ be a component of $\mathcal{U}$ such that $C(A,t_0,n) \subset \mathcal{K}$. Since $\mu^{-1}(t_0)$ is locally connected, $\mathcal{K}$ is open in $\mu^{-1}(t_0)$. Let $K = \bigcup \mathrm{cl}_{\mu^{-1}(t_0)}(\mathcal{K})$. By [10, Lemma 3.1], $K \in C_n(X)$ and by Lemma 3.11, $A \in C(K,\mu(A),n)$.
Let $x_i \in A_i$, for each $i \in \{2,\ldots,s\}$. By [13, Corollary 5.5], there exists a subcontinuum L^* of X such that $A_1 \subset L^*$, $A_1 \neq L^*$ and $L^* \subset N(\gamma,A_1)$. Let $L = \bigcup_{i\neq 1} A_i \cup L$. Notice that $\mu(A) < \mu(L)$. Then $C(A,t_0,n) \subset C(L,t_0) \subset \mathcal{U}$ and $C(L,t_0) \neq C(A,t_0)$. Thus $C(L,t_0) \subset \mathcal{K}$. Let $E \in C(L,t_0) \setminus C(A,t_0)$. Then, $E \cap X \setminus A \neq \emptyset$. Since $E \subset K$, $K \cap X \setminus A \neq \emptyset$. Thus $K \neq A$.
Notice that $K \subset N(\delta,A)$. Now for each $F \in C(K,\mu(A))$, by Lemma 1.2, $H(A,F) < \epsilon$. Hence $C(K,\mu(A)) \subset \mathcal{V}(\epsilon,A)$. Let $\mathcal{G} = \{D \in \mu^{-1}(\mu(A)) : C(D,t_0) \subset \mathcal{K}\}$. Clearly $A \in \mathcal{G}$. By Lemma

3.12, $\mathcal{G}$ is an open subset of $\mu^{-1}([s,t])$. Let $D \in \mathcal{G}$. By Lemma 3.11, $D \in C(K, \mu(A))$. Then $\mathcal{G} \subset C(K, \mu(A))$.
Thus, since $C(K, \mu(A))$ is connected (see [4, Theorem 2.14]) and
$A \in \text{int}(C(K, \mu(A))) \subset C(K, \mu(A)) \subset \mathcal{V}(\epsilon, \mathcal{A})$, we have that $\mu^{-1}([s,t])$ is locally connected. □

Corollary 3.14. *The property of being a locally connected continuum is a increasing stronger size property.*

It is wel known by [4, Theorem 3.1, p. 962] that the property of being an locally connected continuum is a strong size property, so we have the following collorary.

Corollary 3.15. *If X be a locally connected continuum and $0 \leq s < t \leq 1$. Then $\mu^{-1}([s,t])$ is locally connected.*

Lemma 3.16. *Let $K_1, K_2, ..., K_m$ be subcontinua of X. If $K_i \cap K_j = \varnothing$ for each distinct pair $i, j \in \{1, ...m\}$, then the set $< K_1, K_2, ..., K_m >_n \cap \mu^{-1}([s,t])$ is connected for $0 \leq s < t \leq 1$.*

Proof. Put $K = K_1 \cup ... \cup K_m$. Since each K_i is a nonempty set , the definition of $< K_1, K_2, ..., K_m >_n$ implies $K \in < K_1, K_2, ..., K_m >_n$. Let $A, B \in < K_1, K_2, ..., K_m >_n \cap \mu^{-1}([s,t])$, such that $A \neq B$. Suppose that $A \neq K$ and $B \neq K$. By [13, Theorem 1.8, p. 59] there exist order arcs α_1 and α_2 from A to K and from B to K, respectively, contained in $< K_1, K_2, ..., K_m >_n$. If $\mu(K) \in [s,t]$, then we have $< K_1, K_2, ..., K_m >_n \cap \mu^{-1}([s,t])$ is arcwise connected. Suppose $\mu(K) > t$ and consider $t_1, t_2 \in [s,t]$ such that $\mu(\alpha_1(t_1)) = \mu(\alpha_2(t_2)) = t$. Since $< K_1, K_2, ..., K_m >_n \cap \mu^{-1}(t)$ is arcwise connected (see the Proof of [4, Theorem 2.14, p. 960]), there exist an arc from $\mu(\alpha_1(t_1))$ to $\mu(\alpha_2(t_2))$. Thus there exists an arc from A to B. So $< K_1, K_2, ..., K_m >_n \cap \mu^{-1}([s,t])$ is connected. □

Let p, q be distinct points of X. Then X is said to be *aposyndetic at p with respect to q* if there is a subcontinuum K of X such that $p \in int(K)$ and $q \notin K$. If X is aposyndetic at p with respect to every point $q \in X$, then X is said to be *aposyndetic at p*. We say that X is *aposyndetic* if X is aposyndetic at every point of X. We say that X is *finitely aposyndetic*, if for any $p \in X$ and any finite subset F of X such that $p \notin F$, there exists a subcontinuum M of X such that $p \in int(M)$ and $M \cap F = \emptyset$. We say that X is *countable closed set aposyndetic*, if for any $p \in X$ and any countable closed subset F of X such that $p \notin F$, there exists a subcontinuum M of X such that $p \in int(M)$ and $M \cap F = \emptyset$.

Theorem 3.17. *If X is an aposyndetic continuum and $0 \leq s < t \leq 1$, then $\mu^{-1}([s,t])$ is aposyndetic.*

Proof. Let $A, B \in \mu^{-1}([s,t])$ such that $A \neq B$. Let $a \in A - B$. Since X is aposyndetic, for each $x \in B$, there exists a subcontinuum K_x such that $x \in int(K_x)$ and $a \notin K_x$. by compactennes of B, there exist a subset $\{x_1, ..., x_r\} \subset B$ such that $B \subset \bigcup_{j=1}^{r} int(K_{x_j})$ and $a \notin \bigcup_{j=1}^{r} K_{x_j}$. Without loss of generality, we can assume that $K_{x_1}, ..., K_{x_r}$ are pairwise disjoint. By Lemma 3.16 we have that $\langle K_{x_1}, ..., K_{x_r} \rangle \cap \mu^{-1}([s,t])$ is a continuum which is a neighborhood of B in $\mu^{-1}([s,t])$ such that it does not contain A. Thus $\mu^{-1}([s,t])$ is aposyndetic at B with respect to A. Therefore $\mu^{-1}([s,t])$ is aposyndetic. □

Lemma 3.18. *Let $A \in \mu^{-1}([s,t])$ with $0 < s < t \leq 1$. Let $\mathcal{C}$ be a closed subset of $\mu^{-1}([s,t])$ such that $A \notin \mathcal{C}$ and for all $B \in \mathcal{C}$ we have $B \nsubseteq A$ or $A \nsubseteq B$. Then there exist $\epsilon > 0$ such that for all $B \in \mathcal{C}$ we have $B \nsubseteq V_\epsilon(A)$.*

Proof. Suppose the statement is false, then for all $m \in \mathbb{N}$. So, for every $m \in \mathbb{N}$, there exists $B_m \in \mathcal{C}$ such that $B_m \subset \mathcal{V}(1/m, A)$. Since $\mathcal{C}$ is a closed subset of $\mu^{-1}([s,t])$, it is closed in $C_n(X)$. Thus, there exist $B' \in \mathcal{C}$ such that $\{B_m\}_{m=1}^{\infty} \to B'$. And therefore $B' \subset A$, a contradiction. □

Lemma 3.19. *Let $A \in C_n(X)$ such that $\mu(A) \neq 0$. Let $\mathcal{C}$ be a closed subset of $C_n(X)$ such that for all $B \in \mathcal{C}$ we have $B \subsetneq A$ or $A \subsetneq B$. Then there exist $s_0, t_0 \in I$ such that $s_0 < \mu(A) < t_0$ and $\mu^{-1}([s_0, t_0]) \cap \mathcal{C} = \emptyset$. If $A \subsetneq B$ for all $B \in \mathcal{C}$, then there exist $t_0 \in I$ such that $\mu(A) < t_0$ and $\mu^{-1}([\mu(A), t_0]) \cap \mathcal{C} = \emptyset$. And, if $B \subsetneq A$ for all $B \in \mathcal{C}$, then there exist $s_0 \in I$ such that $s_0 < \mu(A)$ and $\mu^{-1}([s_0, \mu(A)]) \cap \mathcal{C} = \emptyset$.*

Proof. If $\mathcal{C}$ is a closed subset of $C_n(X)$, then $\mathcal{C}$ is compact. Thus $\mu(\mathcal{C})$ is a closed subset of I such that $\mu(A) \notin \mu(\mathcal{C})$. □

Theorem 3.20. *Let X be a countable closed set aposyndetic, if $0 < s < t \leq 1$, then $\mu^{-1}([s,t])$ is countable closed set aposyndetic.*

Proof. Let $\mathcal{A}$ be a countable closed subset of $\mu^{-1}([s,t])$. So, $\mu^{-1}([s,t]) - \mathcal{A} \neq \emptyset$. Set $B \in \mu^{-1}([s,t]) - \mathcal{A}$. in order to prove the theorem we will consider 3 cases.

Case 1) Suppose $s < \mu(B) < t$. Let $\mathcal{A}_0 \subset \mathcal{A}$ such that $B \subsetneq C$ or $C \subsetneq B$, for all $C \in \mathcal{A}_0$. By Lemma 3.19 there exist $s_0, t_0 \in I$ such that $s < s_0 < \mu(B) < t_0 < t$ and $\mu^{-1}([s_0, t_0]) \cap \mathcal{A}_0 = \varnothing$. Let $\mathcal{A}_1 = \mathcal{A} \cap \mu^{-1}([s_0, t_0])$. Clearly $\mathcal{A}_1$ is a countable closed subset of $\mu^{-1}([s_0, t_0])$. By Lemma 3.18 there exists $\epsilon > 0$ such that for all $A \in \mathcal{A}_1$ we have $A \not\subseteq \mathcal{V}(\epsilon, B)$. If $U = X - cl_X(\mathcal{V}(\epsilon/2, B))$, then by [7, Theorem 21, p. 58] there exists a map $S : \mathcal{A}_1 \to X$ such that $S(A) \in A \cap U$ for each $A \in \mathcal{A}_1$. Thus $S(\mathcal{A}_1)$ is a countable closed subset of X, such that $S(\mathcal{A}_1) \cap B = \varnothing$. Then, for each $x \in B$, there exists a subcontinuum K_x such that $x \in int_X(K_x) \subset K_x \subset X - S(\mathcal{A}_1)$. Since B is compact, there exist a subset $\{x_1, ..., x_r\} \subset B$ such that $B \subset \bigcup_{j=1}^{r} int(K_{x_j})$. Without loss of generality, we can assume that $K_{x_1}, ..., K_{x_r}$ are pairwise disjoint. By Lemma 3.16 $\langle K_{x_1}, ..., K_{x_r} \rangle \cap \mu^{-1}([s_0, t_0])$ is a subcontinuum of $\mu^{-1}([s_0, t_0])$, such that $B \in int_{C_n(X)}(\langle K_{x_1}, ..., K_{x_r} \rangle) \cap \mu^{-1}([s_0, t_0])$. Since for each $j \in \{1, ..., r\}$, $S(\mathcal{A}_1) \cap K_{x_j} = \emptyset$, we have that $A \not\subseteq \bigcup_{j=1}^{r} K_{x_j}$ for all $A \in \mathcal{A}_1$. Thus $(\langle K_{x_1}, ..., K_{x_r} \rangle \cap \mu^{-1}([s_0, t_0])) \cap \mathcal{A}_1 = \emptyset$ and $(\langle K_{x_1}, ..., K_{x_r} \rangle \cap \mu^{-1}([s_0, t_0])) \cap \mathcal{A} = \varnothing$.

Case 2) Suppose that $\mu(B) = s$. Let $\mathcal{A}_0 \subset \mathcal{A}$ such that $B \subsetneq C$, for all $C \in \mathcal{A}_0$. By Lemma 3.19 there exists $t_0 \in I$ such that $s < t_0 < t$ and $\mu^{-1}([s, t_0]) \cap \mathcal{A}_0 = \varnothing$. Let $\mathcal{A}_1 = \mathcal{A} \cap \mu^{-1}([s, t_0])$. Proceeding in the same way as Case 1) we get subcontinua $K_{x_1}, ..., K_{x_r}$ pairwise disjoint such that $(\langle K_{x_1}, ..., K_{x_r} \rangle \cap \mu^{-1}([s_0, t_0])) \cap \mathcal{A}_1 = \emptyset$ and $(\langle K_{x_1}, ..., K_{x_r} \rangle \cap \mu^{-1}([s_0, t_0])) \cap \mathcal{A} = \varnothing$.

Case 3) Suppose that $\mu(B) = t$. Let $\mathcal{A}_0 \subset \mathcal{A}$ such that $C \subsetneq B$, for all $C \in \mathcal{A}_0$. By Lemma 3.19 there exist $s_0 \in I$ such that $s < s_0 < t$ and $\mu^{-1}([s_0, t]) \cap \mathcal{A}_0 = \varnothing$. Let $\mathcal{A}_1 = \mathcal{A} \cap \mu^{-1}([s_0, t])$. Proceeding in the same way as Case 1), we get subcontinua $K_{x_1}, ..., K_{x_r}$ pairwise disjoint such that $(\langle K_{x_1}, ..., K_{x_r} \rangle \cap \mu^{-1}([s_0, t_0])) \cap \mathcal{A}_1 = \emptyset$ and $(\langle K_{x_1}, ..., K_{x_r} \rangle \cap \mu^{-1}([s_0, t_0])) \cap \mathcal{A} = \emptyset$. □

Corollary 3.21. *Let X be a finitely aposyndetic continuum, if $0 < s < t \leq 1$, then $\mu^{-1}([s,t])$ is finitely aposyndetic.*

4. Acknowledgement

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Discusión

En el objetivo principal que se planteaba en la investigación, se incluía el análisis de ciertas propiedades de un continuo y la relación que existe con los niveles de tamaño fuerte . Esto último para inferir la estructura del n-ésimo hiperespacio de un continuo.

Al realizar la revisión y el análisis de la bibliografía existente se resuelve estudiar la existencia de celdas en los niveles de tamaño fuerte. Al mostrar dicha existencia, se observan ciertas propiedades las cuales pierden el interés en la linea de investigación que se propone en el protocolo.

Para el estudio de las propiedades decrecientes secuenciales de tamaño fuerte se busco una forma para medir la distancia entre elementos de un mismo nivel o de niveles cercanos. Esto nos lleva al estudio de estructuras como los arcos ordenados en el hiperespacio y subcontinuos de los niveles o los bloques.

Conclusiones

Los principales resultandos de la investigación se resumen en los siguientes puntos:

1. La conexidad local, ser un continuo Kelley, la indescomponibilidad, la unicoherencia y encadenabilidad por continuos son propiedades decrecientes secuenciales de tamaño fuerte.

2. La conexidad uniforme por trayectorias, la encadenabilidad uniforme por continuos y la conexidad local son propiedades crecientes de tamaño fuerte.

3. La irreducibilidad no es una propiedad de tamaño fuerte.

4. La aposindesis es una propiedad de bloque de tamaño fuerte

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Anexos

Los resultados de la investigación se presentaron en en los siguientes eventos:

- Platica “Sequential decreasing strong size map“ en el 32 Annual Summer Conference on Topology and Its Applications.
- Platica “Propiedades decrecientes de tamaño fuerte“ en el 50 Congreso Nacional de la Sociedad Matemática Mexicana.
- Platica “$HS_n(X)$ y funciones universales“ en el XI Taller estudiantil de teoría de los continuos y sus hiperespacios.
- Platica “Irreducibilidad en niveles de tamaño fuerte“ en el XIII Taller de continuos, hiperespacios y sistemas dinámicos.
- Platica “Sobre funciones inducidas entre espacios cociente“ en Seminario de los lunes de Topología de UIS.
- Estancia de investigación en la Universidad Industrial de Santander, Colombia.

UNIVERSITY of DAYTON

06/29/2017

To whom this may concern:

This letter is confirming that Miguel Lara Mejia from Universidad Auónoma del Estado de México attended and participated in the Thirty-Second Annual Summer Conference Conference on Topology and Its Applications, held June 26, 2017 through June 30, 2017 at the University of Dayton.

Sincerely

Lynne Yengulalp
Organizer
Thirty-Second Annual Summer Conference Conference on Topology and Its Applications
University of Dayton
300 College Park
Dayton, OH 45469

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PATRICIA J. NAPIER, Notary Public
In and for the State of Ohio
My Commission Expires July 4, 2018

DEPARTMENT OF MATHEMATICS
300 College Park   Dayton, OH 45469-2316
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Bucaramanga, junio 5 de 2018

EL SUSCRITO PROFESOR TITULAR Y DIRECTOR DEL GRUPO DE INVESTIGACIÓN EN MATEMÁTICAS UIS, DE LA UNIVERSIDAD INDUSTRIAL DE SANTANDER

Certifica:

Que el maestro MIGUEL ÁNGEL LARA MEJÍA del Departamento de Matemáticas, Facultad de Ciencias de la Universidad Autónoma del Estado de México ha realizado una estancia en la Escuela de Matemáticas de la Universidad Industrial de Santander, del 20 de febrero al 6 de junio del presente año, donde realizó las siguientes actividades:

1. Presentó en el *Seminario de los lunes de Topología* la conferencia titulada "Sobre funciones inducidas entre espacios cociente", dirigida a estudiantes y profesores de la Escuela de Matemáticas.
2. Participó en reuniones con mi persona, Javier Camargo, donde se discutieron aspectos relacionados con sus proyectos de investigación.
3. Asistió y participó activamente en el curso "Hiperespacios de continuos" que se desarrollo en el primer semestre académico de 2018. Curso de posgrado ofrecido en la Escuela de Matemáticas, con una intensidad de 4 horas por semana, durante 16 semanas.
4. Cada semana asistió al *Seminario de los lunes de Topología* que se realiza en la Escuela de Matemáticas. (http://matematicas.uis.edu.co/jcam/?q=node/6).

Finalmente, considero importante resaltar que el maestro Lara Mejía, asistió a diversas actividades académicas organizadas en la Universidad.

Javier Enrique Camargo García
Profesor Escuela de Matemáticas
Universidad Industrial de Santander

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La Sociedad Matemática Mexicana

otorga el presente

RECONOCIMIENTO

A: **Miguel Angel Lara Mejía**

Por la presentación de la ponencia:
Propiedades decrecientes de tamaño fuerte
realizada dentro de las Actividades del 50 Congreso Nacional de la Sociedad Matemática Mexicana, llevado a cabo del 22 al 27 de Octubre de 2017, en CU, México.

Octubre de 2017
Ciudad de México, México.

Dr. Gelasio Salazar Anaya
Presidente de la SMM

50° CONGRESO NACIONAL DE LA SOCIEDAD MATEMÁTICA MEXICANA

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