UNIVERSIDAD AUTÓNOMA DEL ESTADO DE MÉXICO

FACULTAD DE CIENCIAS

AGUJEROS EN BLOQUES DE WHITNEY EN EL HIPERESPACIO DE SUBCONTINUOS DE DENDRITAS.

TESIS POR ARTÍCULO

QUE PARA OBTENER EL TÍTULO DE:

MAESTRO EN CIENCIAS MATEMATICAS

PRESENTA:

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Índice general

Resumen

Un *continuo* es un espacio métrico, compacto, conexo y no vacío. Sea $C(X)$ el hiperespacio de subcontinuos de un continuo X y $\mu : C(X) \to [0,1]$ una función de Whitney. Sea X un continuo y sea $c, d \in [0,1]$ con $c < d$ y $\mu^{-1}([c,d])$ es unicoherente. Entonces un elemento $A \in \mu^{-1}([c,d])$ *hace un agujero* en $\mu^{-1}([c,d])$ si $\mu^{-1}([c,d]) - \{A\}$ no es unicoherente. En este trabajo se da una caracterización de los elementos $A \in \mu^{-1}([c,d])$ que hacen un agujero en $\mu^{-1}([c,d])$ cuando X es una dendrita .

Introducción

La unicoherencia es una propiedad muy importante en Topología, que es muy útil para distinguir espacios topológicos.
Un continuo Z es *unicoherente* si cada que $Z = A \cup B$, con A y B subconjuntos cerrados y conexos en Z, se tiene que $A \cap B$ es conexo. Una *dendrita* es un continuo localmente conexo y hereditariamente unicoherente. Sean Z un continuo unicoherente y z un elemento de Z, decimos que *z hace un agujero* en Z si $Z - \{z\}$ no es unicoherente, en caso contrario, decimos que z no agujera a Z.
Un *continuo* es un espacio métrico, compacto, conexo y no vacío.
Un *hiperespacio* de un continuo X es una familia de subconjuntos de X que cumplen propiedades especificas. Algunos de ellos son:

$2^X = \{A \subseteq X : A \text{ es cerrado y no vacío}\}$.
$C(X) = \{A \subseteq 2^X : A \text{ es conexo}\}$.

A estos hiperespacios se les considera con la métrica de Hausdorff, los hiperespacios también son continuos.
Recientemente, la clasificación de los puntos que agujeran a un espacio unicoherente ha sido muy útil para distinguir espacios topolgicos, especialmente hiperespacios, por ejemplo: en Lemas 2,1 y 2,2 [**??**, pp. $348 - 349$], el autor muestra que $C_2([0,1]) - A$ es unicoherente para cada $A \in C_2([0,1])$ mientras que $C_2(S^1) - S^1$ no es unicoherente, donde S^1 es la circunferencia unitaria del plano Euclidiano centrada en el origen. Como consecuencia, se obtiene que $C_2([0,1])$ y $C_2(S^1)$ no son homeomorfos en contraste con el hecho que $C([0,1])$ y $C(S^1)$ lo son.
Lo anterior queda plasmado en el siguiente problema :

Problema 1. *Sea X un continuo unicoherente y $\mathcal{H}(X)$ un hiperespacio de X. Determinar que elementos $A \in \mathcal{H}(X)$ agujeran a $\mathcal{H}(X)$.*

Algunas soluciones parciales a este problema se encuentran en [2], [3] y [4].
Una *función de Whitney* para $C(X)$ es una función continua $\mu : C(X) \to [0,1]$ que satisface las siguientes condiciones :
a)$\mu(\{p\}) = 0$ para cada $p \in X$.
b)$\mu(A) < \mu(B)$ siempre que $A \subset B$.
c)$\mu(X) = 1$.
Un *bloque de Whitney* es un conjunto de la forma $\mu^{-1}([a,b])$, para $a, b \in [0,1]$.
Abordaremos este problema en la clase de dendritas y considerando $\mathcal{H}(X)$ como un bloque de Whitney:

Problema 2. *Sea X una dendrita y $\mu : C(X) \to [0,1]$ una función de Whitney para $C(X)$.Determinar que elementos $A \in \mu^{-1}([a,b])$ agujeran a $\mu^{-1}([a,b])$?.*

Protocolo

Agujeros en bloques de Whitney en el hiperespacio de subcontinuos de dendritas

Resumen. El hiperespacio de todos los subcontinuos de un continuo X es denotado por $\mathrm{C}(X)$. Este hiperespacio es considerado con la métrica de Hausdorff. Diremos que un elemento A de $C(X)$ agujera a $C(X)$ si $C(X) - \{A\}$ no es unicoherente. Considerando $\mu: C(X) \to [0,1]$ una función de Whitney para C(X) y $\mu^{-1}([a,b])$ un bloque de Whitney unicoherente con $0 \leq a < b \leq 1$ el presente proyecto está enfocado a caracterizar los elementos A de $\mu^{-1}([a,b])$ tales que A agujera a $\mu^{-1}([a,b])$ cuando X es una dendrita.

1. ANTECEDENTES

La Teoría de Hiperespacios, una rama importante de la topología, tuvo sus inicios a principios del Siglo XX y desde entonces la investigación en esta área ha experimentado un interés creciente. La teoría de hiperespacios se encarga de estudiar familias particulares de subconjuntos de los espacios topológicos. Esta teoría ha mostrado ser muy útil en el comportamiento topológico de los espacios originales con respecto a las propiedades que presentan los hiperespacios y viceversa, lo cual se refleja en la amplia bibliografía que existe al respecto. Algunos de los hiperespacios más estudiados para un espacio métrico X son:

$$CL(X) = \{A \subseteq X: A \text{ es cerrado en } X \text{ y } A \neq \emptyset\},$$
$$2^X = \{A \in CL(X): A \text{ es compacto}\},$$
$$C(X) = \{A \in 2^X: A \text{ es conexo}\},$$
$$F_n(X) = \{ A \in 2^X: A \text{ tiene a lo más } n \text{ puntos}\} \text{ y}$$
$$C_n(X) = \{ A \in 2^X: A \text{ tiene a lo más } n \text{ componentes}\}.$$

Un *continuo* es un espacio métrico, compacto, conexo y no vacío. Un continuo es una dendrita si es localmente conexo y no contiene circunferencias. Una función de Whitney para C(X), es una función continua $\mu: C(X) \to [0,1]$ que satisface las siguientes condiciones:
a) $\mu(\{p\}) = 0$ para cada $p \in X$
b)$\mu(A) < \mu(B)$ siempre que $A \subset B$
c)$\mu(X) = 1$
Dados $a, b \in [0,1]$ un bloque de Whitney es el conjunto $\mu^{-1}([a,b])$.

Hasta el momento, los hiperespacios para un continuo X más investigados han sido 2^X y $C(X)$. A pesar que se tiene un amplio conocimiento sobre ambos hiperespacios, aún existe terreno poco explorado. Por ejemplo, caracterizar los elementos A de $C(X)$ tal que A agujera a $C(X)$. El presente proyecto está enfocado a estudiar este problema para la clase de dendritas y considerando a un bloque de Whitney en vez de considerar a todo el hiperespacio de subcontinuos.

2. JUSTIFICACIÓN

Un espacio topológico X es *unicoherente* si para cada par de subconjuntos cerrados y conexos de K y L de X tales que $X = K \cup L$ se cumple que $K \cap L$ es conexo. Un punto x de un espacio unicoherente X *agujera a* X si $X - \{x\}$ no es unicoherente. La clasificación de los puntos que agujeran a un espacio unicoherente ha sido muy útil para distinguir espacios topológicos, especialmente hiperespacios, por ejemplo: en Lemas 2.1 y 2.2 [1, pp. 348-349], el autor muestra que $C_2([0,1]) - \{A\}$ es unicoherente para cada $A \in C_2([0,1])$ mientras que $C_2(S^1) - \{S^1\}$ no es unicoherente, donde S^1 es la circunferencia unitaria del plano Euclidiano centrada en el origen. Como consecuencia, se obtiene que $C_2([0,1])$ y $C_2(S^1)$ no son homeomorfos en contraste con el hecho que $C([0,1])$ y $C(S^1)$ lo son.

El presente proyecto está enfocado a obtener una clasificación de puntos que agujeran a un bloque de Whitney del hiperespacio de subcontinuos de un continuo, en la familia de dendritas, con la finalidad de contribuir a hacer más completo el entendimiento sobre las relaciones entre los hiperespacios y el continuo base.

3. DEFINICIÓN DEL PROBLEMA

Estamos interesados en el siguiente problema que aparece por primera vez en [2, p. 2000]:

PROBLEMA 1: Dado un continuo X, caracterizar los elementos A de $H(X)$ tal que A agujera a $H(X)$.

En este proyecto daremos otro enfoque relacionado a este problema, estudiaremos los bloques de Whitney en lugar de estudiar todo el espacio, en el siguiente sentido, dada una dendrita X, $\mu: C(X) \to [0,1]$ una función de Whitney, ¿para cuales elementos $A \in \mu^{-1}([a,b])$ se tiene que A agujera a $\mu^{-1}([a,b])$

Soluciones parciales al Problema 1 pueden ser consultadas en [2], [3], [4], [5], [6], [7] y [8]

4. OBJETIVOS Y METAS

Dar una caracterización de los elementos A de $\mu^{-1}([a,b])$ que agujeran a $\mu^{-1}([a,b])$ cuando X es una dendrita y $\mu: C(X) \to [0,1]$ una función de Whitney.

5. METODOLOGÍA

Se empleará la metodología usual en proyectos de investigación en matemáticas:

- Se realizará discusiones conjuntas con los Tutores Académicos sobre los avances conseguidos.
- Se expondrán artículos relacionados al tema de investigación ante los Tutores Académicos.
- Se realizará investigación personal.
- Se realizará investigación bibliográfica existente con fines de generar nuevas líneas de investigación.
- Se expondrán los resultados obtenidos más importantes en congresos nacionales o talleres.
- Se participará activamente en las sesiones semanales del Seminario Permanente de Hiperespacio de Continuos de la Facultad de Ciencias de la UAEMéx.

6. PRODUCTOS COMPROMETIDOS

Envío de un artículo de investigación que contenga los resultados obtenidos más importantes al proceso de arbitraje de una revista indizada, de circulación internacional y especializada en el área para su posible publicación.

7. PROGRAMA DE ACTIVIDADES CALENDARIZADO

Actividades	**Semestre**			
	1ero	2do	3er	4to
ALGEBRA MODERNA	X			
ANALISIS REAL Y COMPLEJO I	X			
SEMINARIO INTERDISCIPLINARIO I	X			
ACTIVIDADES DE INVESTIGACION DE MAESTRIA I	X			
HIPERESPACIOS DE CONTINUOS		X		
TEMAS SELECTOS DE TEORIA DE CONTINUOS		X		
SEMINARIO INTERDISCIPLINARIO II		X		
ACTIVIDADES DE INVESTIGACION DE MAESTRIA II		X		
TOPOLOGIA I			X	
TOPOLOGIA II			X	
SEMINARIO INTERDISCIPLINARIO III			X	
ACTIVIDADES DE INVESTIGACION DE MAESTRIA III			X	
SEMINARIO INTERDISCIPLINARIO IV				X
ACTIVIDADES DE INVESTIGACION DE MAESTRIA IV				X
ASISTENCIA Y PARTICIPACIÓN EN EL SEMINARIO PERMANENTE DE TEORÍA DE CONTINUOS Y SUS HIPERESPACIOS	X	X	X	X

INVESTIGACIÓN CONJUNTA CON TUTORES ACADÉMICOS	X	X	X	X
INVESTIGACIÓN PERSONAL	X	X	X	X
INVESTIGACIÓN BIBLIOGRÁFICA Y EXPOSICIÓN DE ARTÍCULOS	X	X	X	X
REDACCIÓN DE LOS RESULTADOS OBTENIDOS MÁS IMPORTANTES			X	X
ENVÍO DE ARTÍCULO DE INVESTIGACIÓN				X
OBTENCIÓN DEL GRADO				

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Artículo

MAKING HOLES IN WHITNEY BLOCKS IN THE HYPERSPACE OF DENDRITES

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Abstract. A *continuum* is a non-empty, compact, connected, metric space. Let $\mathcal{C}(X)$ be the hyperspace of all subcontinua of a continuum X and let $\mu : \mathcal{C}(X) \to [0,1]$ be a Whitney map. Let X be a continuum and let $c, d \in [0,1]$ be such that $c < d$ and $\mu^{-1}([c,d])$ is unicoherent. Then an element $A \in \mu^{-1}([c,d])$ *makes a hole* in $\mu^{-1}([c,d])$ if $\mu^{-1}([c,d]) - \{A\}$ is not unicoherent. In this paper, the elements $A \in \mu^{-1}([c,d])$ satisfying A makes a hole in $\mu^{-1}([c,d])$ when X is a dendrite are characterized.

1. Introduction

A connected topological space Z is *unicoherent* provided that $A \cap B$ is connected whenever A and B are connected closed subsets of Z such that $Z = A \cup B$. A point z in a unicoherent topological space Z *makes a hole* in Z if $Z - \{z\}$ is not unicoherent.

A *continuum* is a non-empty, compact, connected, metric space. The hyperspace of all subcontinua of a continuum X is denoted by $\mathcal{C}(X)$ endowed with the Hausdorff metric (see [15, p. 53] for the definition of Hausdorff metric). A *Whitney map* for $C(X)$ is a map $\mu : C(X) \to [0,1]$ such that:

(μ.1) $\mu(\{p\}) = 0$ for every $p \in X$.

(μ.2) $\mu(A) < \mu(B)$ whenever $A \subsetneq B$.

(μ.3) $\mu(X) = 1$.

Whitney maps always exist for every continuum (see [13, Theorem 13.4, p. 107]). A *Whtiney block* in $\mathcal{C}(X)$ is a set of the form $\mu^{-1}([c,d])$ where $0 \le c < d \le 1$.

A dendrite is a hereditarily unicoherent locally connected continuum (*hereditarily unicoherent* means each one of its subcontinua is unicoherent).

The main interest of this paper is the following problem arised in [1, p. 2000].

Problem. Let $\mathcal{H}$ be a unicoherent hyperspace of a continuum X. For which elements $A \in \mathcal{H}$, A makes a hole in $\mathcal{H}$.

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Key words and phrases. continuum, make a hole, Whitney map, Whitney block, unicoherence, property b), dendrite.

The unicoherence of all Whitney block in the hyperspace of a dendrite is proved in Theorem 2.11. In the current paper, the partial solution to this problem when X is a dendrite and $\mathcal{H}$ is a Whitney block in $\mathcal{C}(X)$ is presented.

Readers specially interested in this problem are referred to [2–7].

2. PRELIMINARIES AND AUXILIARY RESULTS

The set of all positive integer will be denoted by $\mathbb{N}$.

Let X be a topological space. Given a subset A of X, the closure of A and the interior of A in X are represented by $\mathrm{Cl}_X A$ and $\mathrm{Int}_X A$, respectively. A point $x \in X$ is a *cut point* of X provided that $X - \{x\}$ is not connected. The interval $[0,1]$ is denoted by I. An *arc* is any homeomorphic space to I.

Let X be a dendrite. If z is a point of X, then by *order of z in X*, denoted by $\mathrm{ord}(z, X)$, is understood the Menger-Uryshon order (see [15, Definition 9.3, p. 141]), or equivalently, the order in the classical sense, this means the number of arcs emanating from z and disjoint out of z (see [8, p. 229]). The symbol $E(X)$ denotes the set $\{x \in X : \mathrm{ord}(x, X) = 1\}$. For $p, q \in X$, the symbol pq denotes the unique arc contained in X between p and q if $p \neq q$, and $pq = \{p\}$ otherwise.

A *map* stands for a continuous function between topological spaces. Denote by exp the map from the real line $\mathbb{R}$ onto the unit circle in the Euclidean plane centered at the origin S^1 defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. A map f from a connected topological space Z into S^1 *has a lifting* if there exists a map h from $\mathbb{R}$ into S^1 such that $f = \exp \circ h$. A connected topological space Z has *property* b) if each map $f : Z \to S^1$ has a lifting. In locally connected, connected metric space, to have property $b)$ and unicoherence are equivalent (see [10, Theorem 3, p. 70]).

A subset Z of a topological space W is a *deformation retract* of W if there exists a map $L : W \times I \to W$ such that $L(x, 0) = x$ for every $x \in W$, $L(X \times \{1\}) = Z$ and $L(z, 1) = z$ for each $z \in Z$. Such homotopy L is called *deformation retraction* from W to Z.

Proposition 2.1. *Let X be a locally connected unicoherent continuum, let μ be a Whitney map for $\mathcal{C}(X)$ and let $c, d \in I$ be such that $c < d$. If $\mathcal{B}$ is a connected open subset of $\mu^{-1}([c, d])$ and $l \in [c, d]$ are such that $\mu^{-1}(l)$ is a deformation retract of $\mathcal{B}$, then $\mathcal{B}$ has property b).*

Proof. In light of [12, Theorem A, p. 252], $\mu^{-1}(l)$ is unicoherent. From our assumption $\mu^{-1}(l)$ is deformation retract of $\mathcal{B}$ and [9, Theorem 3, p. 160], it follows that $\mathcal{B}$ is unicoherent.

Now, invoke [1, Theorem 3.1] to conclude that $\mu^{-1}([c,d])$ is locally connected. Now, this and the fact that $\mathcal{B}$ is an open subset of $\mu^{-1}([c,d])$ together imply that $\mathcal{B}$ is locally connected.

In conclusion, $\mathcal{B}$ is a unicoherent, connected, locally connected, metric space and so, by [10, Theorem 3, p. 70], $\mathcal{B}$ has property b). □

Proposition 2.2. *Let X be a continuum and let $c, d, u, v, w \in I$ be such that $u \in I - [c,d]$, $\{v, w\} = \{c, d\}$ and $|v - u| = \min\{|c-u|, |d-u|\}$. If $L : \mathcal{C}(X) \times I \to \mathcal{C}(X)$ is a map such that each one of the following statements holds:*

- $(L.1)$ *$L(B, 0) = B$ for each $B \in \mathcal{C}(X)$,*
- $(L.2)$ *$L(\mathcal{C}(X) \times \{1\}) \subseteq \mu^{-1}(u)$,*
- $(L.3)$ *$L(B, s) \subsetneq L(B, t)$ whenever $s, t \in I$ are such that $s < t$ and $\mu(B) < u$ and*
- $(L.4)$ *$L(B, t) \subsetneq L(B, s)$ whenever $s, t \in I$ are such that $s < t$ and $\mu(B) > u$,*

then there exists a map $\varphi_L : \mu^{-1}([c,d]) \times I \to \mu^{-1}([c,d])$ satisfying:

- $(\varphi.1)$ *$\varphi_L(D, 0) = D$ for each $D \in \mu^{-1}([c,d])$,*
- $(\varphi.2)$ *$\varphi_L(\mu^{-1}([c,d]) \times \{1\}) = \mu^{-1}(v)$,*
- $(\varphi.3)$ *$\varphi_L^{-1}(D) = \{(D, 0)\}$ for each $D \in \mu^{-1}(w)$, and*
- $(\varphi.4)$ *$\varphi_L(M, 1) = M$ for each $M \in \mu^{-1}(v)$.*

Proof. Let $B \in \mu^{-1}([c,d])$. Condition $(L.1)$ and $(L.2)$ and the assumptions on u and v guarantees that $\{L(B,t) : t \in I\} \cap \mu^{-1}(v)$ must be non-empty. From this, $(L.3)$ and $(L.4)$, it follows that there exists a unique $t_B \in I$ such that $\mu(L(B, t_B)) = v$. Define the function $\theta : \mu^{-1}([c,d]) \to I$ by $\theta(B)$ is the unique element in I satisfying that $\mu(L(B, \theta(B))) = v$. Let us show that θ is a map.

Let $\{B_n\}_{n=1}^{\infty}$ be a sequence in $\mu^{-1}([c,d])$ converging to $B_0 \in \mu^{-1}([c,d])$. The compactness of I allows us assume that the sequence $\{\theta(B_n)\}_{n=1}^{\infty}$ converges to some $q \in I$. Apply the continuity of L and of μ to get the equality $\mu(L(B_0, q)) = \lim \mu(L(B_n, \theta(B_n))) = v$. Hence, $\theta(B_0) = q$. This proves that θ is continuous.

Define $\varphi_L : \mu^{-1}([c,d]) \times I \to \mu^{-1}([c,d])$ by letting

$$\varphi_L(B, t) = L(B, t\theta(B))).$$

In order to see that φ_L is well defined, let $(D,t) \in \mu^{-1}([c,d]) \times I$. If $\mu(D) < u$, then the inclusions $L(D,\theta(D)) \subseteq \varphi_L(D,t) \subseteq D$ are implied by $(L.3)$ and so $\mu(\varphi_L(D,t)) \in [c,d]$. Under the assumption $\mu(D) > u$, condition $(L.4)$ guarantees that $D \subseteq \varphi_L(D,t) \subseteq L(D,\theta(D))$ and this implies that $\varphi(D,t) \in [c,d]$. This proves that φ_L is well defined.

The continuity of φ_L follows from so are L and θ. From the definition of φ_L and $(L.1)$, it follows that $(\varphi.1)$ holds. Now, let $D \in \mu^{-1}(w)$. The inclusion $(D,0) \in \varphi_L^{-1}(D)$ is an immediately consequence of condition $(L.1)$. In order to see that $\varphi_L^{-1}(D)$ is contained in $\{(D,0)\}$, let $(K,s) \in \mu^{-1}([c,d])$ be such that $\varphi_L(K,s) = D$. Let us prove that $s = 0$. Seeking a contradiction assume that $s > 0$. In light of $(L.1)$ and $(L.3)$, if $\mu(K) < u$, then $v = d$, $w = c$ and $K = L(K,0) \subsetneq L(K,s\theta(K)) = \varphi_L(K,s) = D$ and so $\mu(K) < \mu(D) = c$. Hence, $\mu(K) > u$. Thus, $v = c$ and $w = d$. By $(L.4)$, the inclusions $D = \varphi_L(K,s) = L(K,s\theta(K)) \subsetneq L(K,0) = K$ holds. Then $d = \mu(D) < \mu(K)$. A contradiction. This implies that $s = 0$ and so, by $(\varphi.1)$, $D = \varphi_L(K,0) = K$. Therefore, $\varphi_L^{-1}(D) = \{(D,0)\}$.

Finallly, notice that if $D \in \mu^{-1}([c,d])$, then $\mu(\varphi_L(D,1)) = \mu(L(D,\theta(D))) = v$, and if $M \in \mu^{-1}(v)$, then $\theta(M) = 0$ and so, by $(L.1)$, $\varphi_L(M,1) = L(M,\theta(M)) = L(M,0) = M$. Thus, $(\varphi.2)$ and $(\varphi.4)$ are true. □

The next result will be used without mention it explicitly throughout this paper.

Proposition 2.3. *Let X be a continuum, let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}(X)$ and let $B \in \mathcal{C}(X)$. Then $B = \lim B_n$ if and only if each one of the following statements holds.*

1. *For each $x \in B$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim x_n = x$ and each $x_n \in B_n$.*
2. *If $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence in $\mathbb{N}$ and $\{y_k\}_{k\in\mathbb{N}}$ is a convergent sequence in X such that $y_k \in B_{n_k}$ for each $k \in \mathbb{N}$, then $\lim y_k \in B$.*

Proposition 2.4. *Let X be a dendrite and let $a, b \in X$. If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences in X such that $\lim a_n = a$ and $\lim b_n = b$, then $\lim a_nb_n = ab$.*

Proof. First, assume that $a = b$. Observe that $a \in \lim a_nb_n$. In order to see that $\lim a_nb_n \subseteq \{a\}$, let $y \in X - \{a\}$. Then, there exists a connected open subset V of X such that $a \in V \subseteq \mathrm{Cl}_X V \subseteq X - \{y\}$. Now, there exists $m \in \mathbb{N}$ such that $a_n, b_n \in V$ for each $n \geq m$. On the other hand, by [15, Theorem 8.26, p. 132], the set V is arcwise connected. Hence, the uniqueness of the arcs in dendrites implies that $a_nb_n \subseteq V$ for each $n \geq m$. Thus, if

$\{z_n\}_{n\in\mathbb{N}}$ is a convergent sequence of points of X such that each $z_n \in a_nb_n$, then $\lim z_n \in \mathrm{Cl}_X V$ and so $y \notin \lim a_nb_n$. In conclusion, $\lim a_nb_n = \{a\} = ab$.

Second, assume that $a \neq b$. In light of [15, Theorem 10.2, p. 166], there exists $p \in X$ and non-empty disjoint open subsets U and V of X such that $X \setminus \{p\} = U \cup V$, $a \in U$ and $b \in V$. Let $m \in \mathbb{N}$ be such that $a_n \in U$ and $b_n \in V$ for each $n \geq m$. Notice that if $n \geq m$, then $p \in a_nb_n$. Now, [14, Corollary 4, p. 298] guarantees that X is arc-smooth at p. This implies that $\lim pa_n = pa$ and $\lim pb_n = pb$ and so $\lim a_nb_n = \lim(a_np \cup b_np) = (\lim pa_n) \cup (\lim pb_n) = pa \cup pb = ab$. The proof is complete. □

Given a proper subcontinuum A of a dendrite X, for each $x \in X - A$, by [15, Lemma 10.24, p. 175], there exists a unique point $\psi_A(x) \in A$ such that $\psi_A(x)$ is a point of any arc in X from x to any point of A. Define $r : X \times C(X) \to X$ by letting $r(x, A) = \psi_A(x)$ if $x \in X - A$ and letting $r(x, A) = x$ if $x \in A$.

Proposition 2.5. *The function r is a map.*

Proof. Let $\{(x_n, A_n)\}$ be a sequence in $X \times \mathcal{C}(X)$ converging to $(x, A) \in X \times \mathcal{C}(X)$. The compactness of X allows us to assume that there exists $y \in X$ such that $\lim r(x_n, A_n) = y$. Since $\lim A_n = A$ and each $r(x_n, A_n) \in A_n$, the inclusion $y \in A$ holds. Now, let us prove that $y \in xa$ for each $a \in A$. Choose $a \in A$. Then there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X such that $a_n \in A_n$ for each $n \in \mathbb{N}$ and $\lim a_n = a$. Apply Proposition 2.4 to obtain that $\lim x_na_n = xa$. Now, from this and the fact that $r(x_n, A_n) \in x_na_n$ for each $n \in \mathbb{N}$, it follows that $y = \lim r(x_n, A_n) \in \lim x_na_n = xa$. Thus, $r(x, A) = y$. □

Lemma 2.6. *Let X be a dendrite, let $a, b \in X$ and let μ be a Whitney map for $\mathcal{C}(X)$. If $0 \leq t \leq \mu(ab)$, then there exists a unique element $z \in ab$ such that $\mu(az) = t$.*

Proof. Let $g : ab \to [0, \mu(a, b)]$ be the function defined by $g(x) = \mu(ax)$. Let us prove that g is a homeomorphism.

In order to see that g is one-to-one, let $x, y \in ab$ be such that $g(x) = g(y)$. Observe that either $ax \subseteq ay$ or $ay \subseteq ax$. From the fact that $\mu(ax) = \mu(ay)$, it follows that $ax = ay$. Then $x = y$.

Now, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in ab converging to $x_0 \in ab$. In light of Proposition 2.4, the sequence $\{ax_n\}_{n\in\mathbb{N}}$ converges to ax_0. The continuity of μ guarantees that $\lim g(x_n) = \lim \mu(ax_n) = \mu(ax_0) = g(x_0)$. In conclusion, g is continuous.

Finally, observe that $g(ab)$ is a connected subset of $[0, \mu(ab)]$, $g(a) = 0$ and $g(b) = \mu(ab)$. Hence, g must be an onto map.

Therefore, g is a homeomorphism. $\square$

Lemma 2.7. *Let $x \in X$ and let $A \in C(X)$. If $y \in r(x, A)x$, then $r(y, A) = r(x, A)$.*

Proof. First, assume that $y \in A$. Then $r(x, A) \in xy$ and $r(y, A) = y$. From this and our assumption follows that $xy \subseteq r(x, A)x$ and $r(x, A)x \subseteq xy$. Hence, $xy = r(x, A)x$ and so $r(y, A) = y = r(x, it, A)$.

Suppose that $y \in X \setminus A$. Since $r(x, A) \in A$, $r(y, A) \in r(x, A)y \subseteq r(x, A)x$. Now, observe that if $a \in A$, then $r(y, A) \in r(x, A)x \subseteq ax$. So, $r(y, A) \in A$ belonging to any arc in X from x to any point of A. This implies that $r(x, A) = r(y, A)$. $\square$

For a dendrite X and a Whitney map $\mu : \mathcal{C}(X) \to I$, define $\lambda : X \times X \times \mathcal{C}(X) \times I \to X$ by $\lambda(x, y, A, t)$ is the unique element of $r(y, A)x$ such that $\mu(r(y, A)\lambda(x, y, A, t)) = t\mu(r(y, A)x)$.

Proposition 2.8. *The function λ is a map and, for each $(x, y, A) \in X \times X \times \mathcal{C}(X)$, each one of the following statements holds.*

(λ.1) *$\lambda(x, y, A, 1) = x$.*
(λ.2) *$\lambda(x, y, A, 0) = r(y, A)$. and*
(λ.3) *If $t, s \in I$, then $\lambda(\lambda(x, y, A, s), y, A, t) = \lambda(x, y, A, ts)$.*

Proof. First, the fact that λ is well defined follows from Lemma 2.6. In order to prove the continuity of λ, let $\{(x_n, y_n, A_n, t_n)\}_{n\in\mathbb{N}}$ be a sequence in $X \times X \times C(X) \times I$ converging to $(x, y, A, t) \in X \times X \times C(X) \times I$. The compactness of X allows us to suppose that there exists $z \in X$ such that $\lim \lambda(x_n, y_n, A_n, t_n) = z$. Let us prove that $z \in r(y, A)x$ and $\mu(r(y, A)z) = t\mu(r(y, A)x)$.

Observe that Propositions 2.4 and 2.5 together guarantee that $\lim r(y_n, A_n)x_n = r(y, A)x$ and $\lim r(y_n, A_n)\lambda(x_n, y_n, A_n, t_n) = r(y, A)z$. Now, since each $\lambda(x_n, y_n, A_n, t_n) \in r(y_n, A_n)x_n$, the inclusion $z \in r(y, A)x$ holds. The continuity of μ implies that $\mu(r(y, A)z) = \lim \mu(r(y_n, A_n)\lambda(x_n, y_n, A_n, t_n)) = \lim t_n\mu(r(y_n, A_n)x_n) = t\mu(r(y, A)x)$. Hence, $z = \lambda(x, y, A, t)$. In conclusion, λ is a map.

Next, by definition, $\lambda(x, y, A, 0), \lambda(x, y, A, 1) \in r(y, A)x$ are such that $\mu(r(y, A)\lambda(x, y, A, 0)) = 0$ and $\mu(r(y, A)\lambda(x, y, A, 1)) = \mu(r(y, A)x)$ and so, $\lambda(x, y, A, 0) = r(y, A)$ and $\lambda(x, y, A, 1) = x$. This proves that (λ.1) and (λ.2) are satisfied.

Finally, by definition, $\lambda(x, y, A, s) \in r(y, A)x$ and $\lambda(\lambda(x, y, A, s), y, A, t) \in r(y, A)\lambda(x, y, A, s)$ are such that $\mu(r(y, A)\lambda(x, y, A, s)) = s\mu(r(y, A)x)$ and

$\mu(r(y,A)\lambda(\lambda(x,y,A,s),y,A,t)) = t\mu(r(y,A)\lambda(x,y,A,s))$. So, $ts\mu(r(y,A)x) = \mu(r(y,A)\lambda(\lambda(x,y,A,s),y,A,t))$ and $r(y,A)\lambda(x,y,A,s) \subseteq r(y,A)x$. This implies that $\lambda(\lambda(x,y,A,s),y,A,t) = \lambda(x,y,A,ts)$. Thus, λ fulfills (λ.3). $\square$

Let $p \in X$. Define $F : \mathcal{C}(X) \times I \to \mathcal{C}(X)$ by

$$F(A,t) = \lambda(A \times \{p\} \times \{A\} \times \{t\}).$$

Theorem 2.9. *The function F is a map such that each one of the following statements holds.*

- $(F.1)$ *$F(A,0) = A$ for each $A \in \mathcal{C}(X)$.*
- $(F.2)$ *$F(\mathcal{C}(X) \times \{1\}) = \mu^{-1}(0)$.*
- $(F.3)$ *If $A \in \mathcal{C}(X)$ and $s,t \in I$ are such that $s < t$, then $F(A,t) \subsetneq F(A,s)$.*
- $(F.4)$ *$F(\{x\},1) = \{x\}$ for each $x \in X$.*

Proof. The continuity of λ implies that F is a map. Invoke (λ.1) and (λ.2) to see that

$$F(A,1) = \{\lambda(x,p,A,0) : x \in A\} = A$$

and

$$F(A,0) = \{\lambda(x,p,A,1) : x \in A\} = \{r(p,A)\} \in \mu^{-1}(0)$$

for each $A \in \mathcal{C}(X)$. Hence, $(F.1)$ and $(F.2)$ are satisfied.

Since $r(p,\{x\}) = x$ for each $x \in X$, the condition $(F.4)$ holds.

Next, to show that F fulfils $(F.3)$, let $A \in \mathcal{C}(X)$ and let $s,t \in I$ be such that $s < t$. Take $y \in F(A,s)$. This means there exists $x \in A$ such that $y = \lambda(x,p,A,s)$. In light of (λ.3), $y = \lambda(\lambda(x,p,A,\frac{s}{t}),p,A,t)$. So, since $\lambda(x,p,A,\frac{s}{t}) \in r(p,A)x \subseteq A$, $y \in F(A,t)$. This proves that $F(A,s) \subseteq F(A,t)$. Finally, the compactness of A guarantees that there exists $q \in A$ such that $\mu(r(p,A)q) = \sup\{\mu(r(p,A)z) : z \in A\}$. Let us argue that $\lambda(q,p,A,t) \in F(A,t) \setminus F(A,s)$. Seeking a contradiction, assume that there exists $w \in A$ such that $\lambda(w,p,A,s) = \lambda(q,p,A,t)$. From the fact that $\mu(r(p,A)\lambda(w,p,A,s)) = s\mu(r(p,A)w)$ and $\mu(r(p,A)\lambda(q,p,A,t)) = t\mu(r(p,A)q)$, it follows that $\mu(r(p,A)w) = \frac{t}{s}\mu(r(p,A)q) > \mu(r(p,A)q)$. This contradicts the choice of q. In conclusion, $F(A,s)$ is a proper subset of $F(A,t)$. $\square$

Define $G : \mathcal{C}(X) \times I \to \mathcal{C}(X)$ by

$$G(A,t) = \{\lambda(x,x,A,t) : x \in X\}.$$

Proposition 2.10. *The function G is a map such that each one of the following statements holds.*

$(G.1)$ *$G(A,0)=A$ for each $A\in\mathcal{C}(X)$.*
$(G.2)$ *$G(\mathcal{C}(X)\times\{1\})=\{X\}$.*
$(G.3)$ *If $A\in\mathcal{C}(X)$ and $s,t\in I$ are such that $s<t$, then $G(A,s)\subsetneq G(A,t)$.*

Proof. The continuity of G follows from so is λ. Now, let $A\in\mathcal{C}(X)$. Apply $(\lambda.1)$ and $(\lambda.2)$ to get the equalities

$$G(A,0)=\{\lambda(x,x,A,0):x\in X\}=\{r(x,A):x\in X\}=A$$

and

$$G(A,1)=\{\lambda(x,x,A,1):x\in X\}=X.$$

So, $(G.2)$ and $(G.3)$ are satisfied.

Now, let $A\in\mathcal{C}(X)$ and let $s,t\in I$ be such that $s<t$. If $s=0$, then $G(A,0)=A$, $\lambda(a,a,A,t)=a$ for each $a\in A$ and $\lambda(x,x,A,t)\in X\setminus A$ for each $x\in X\setminus A$ and so $G(A,0)$ is a proper subset of $G(A,t)$. Assume that $s>0$ and take $y\in G(A,s)$. Then there exists $x\in X$ such that $y=\lambda(x,x,A,s)$. By Lemma 2.6, there exists $w\in r(x,A)x$ fulfilling $\mu(r(x,A)w)=\frac{s}{t}\mu(r(x,A)x)$. Lemma 2.7 guarantees that $r(w,A)=r(x,A)$. Then, $\mu(r(w,A)y)=\mu(r(x,A)y)=s\mu(r(x,A)x)=t\mu(r(w,A)w)$ and so $y=\lambda(w,w,A,t)\in G(A,t)$. Finally, let us argue that $G(A,s)$ is a proper subset of $G(A,t)$. The compactness of X guarantees that there exists $z\in X$ such that $\mu(r(z,A)z)=\sup\{\mu(r(b,A)b):b\in X\}$. Set $q=\lambda(z,z,A,t)$. Thus, $q\in G(A,t)$, $q\in r(z,A)z$, $\mu(r(z,A)q)=t\mu(r(z,A)z)$ and $r(z,A)=r(q,A)$. If q were an element of $G(A,s)$, then there would exist $u\in X$ such that $q=\lambda(u,u,A,s)$ and so $q\in r(u,A)u$, $r(q,A)=r(u,A)$ and $\mu(r(u,A)u)=\frac{1}{s}\mu(r(u,A)q)=\frac{1}{s}\mu(r(q,A)q)=\frac{1}{s}\mu(r(z,A)q)=\frac{t}{s}\mu(r(z,A)z)>\mu(r(z,A)z)$, contradicting the choice of z. Therefore, $G(A,s)$ is a proper subset of $G(A,t)$. □

Theorem 2.11. *Let X be a dendrite, let μ be a Whitney map for $C(X)$ and let $c,d\in I$ be such that $c<d$. Then $\mu^{-1}([c,d])$ is unicoherent.*

Proof. Let us start by proving that $\mu^{-1}(c)$ is a deformation retract of $\mu^{-1}([c,d])$. If $c=0$, then $(F.1)$, $(F.2)$ and $(F.3)$ together shows that $F|_{\mu^{-1}([c,d])\times I}:\mu^{-1}([c,d])\times I\to\mu^{-1}([c,d])$ is deformation retraction of $\mu^{-1}([c,d])$ to $\mu^{-1}(c)$. Now, assume that $c>0$.

Proposition 2.2 guarantees that there exists a map $\varphi_F:\mu^{-1}([c,d])\times I\to\mu^{-1}([c,d])$ satisfying that $\varphi_F(A,0)=A$ and $\varphi_F(A,1)\in\mu^{-1}(c)$ for each $A\in\mu^{-1}([c,d])$, and $\varphi_F(B,1)=B$ for each $B\in\mu^{-1}(c)$. Then $\mu^{-1}(c)$ is a deformation retract of $\mu^{-1}([c,d])$.

Now, from this and Proposition 2.1, it follows that $\mu^{-1}([c,d])$ has property *b*). Apply [10, Theorem 2, p. 69] to get that $\mu^{-1}([c,d])$ is unicoherent. □

Theorem 2.12. *Let X be a dendrite, let μ be a Whitney map for $C(X)$ and let $c, d \in I$ be such that $c < d$. Each one of the following conditions holds.*

(2.12.1) *If $A \in \mu^{-1}(c)$, then $\mu^{-1}([c,d]) - \{A\}$ has property b).*
(2.12.2) *If $A \in \mu^{-1}(d)$, then $\mu^{-1}([c,d]) - \{A\}$ has property b).*

Proof. Let us prove (2.12.1). Assume that $d = 1$. From $(G.1)$ and $(G.3)$, it follows that $G^{-1}(A) \subseteq \{K \in \mathcal{C}(X) : K \subseteq A\} \times I$. This and conditions $(G.1)$, $(G.2)$ and $(G.3)$ together shows that the restriction map $G|_{(\mu^{-1}([c,d])-\{A\})\times I} : (\mu^{-1}([c,d]) - \{A\}) \times I \to \mu^{-1}([c,d]) - \{A\}$ is a deformation retract from $\mu^{-1}([c,d]) - \{A\}$ to $\mu^{-1}(d)$. Now, suppose that $d < 1$. Taking $u = 1$, by $(G.1)$, $(G.2)$ and $(G.3)$, G satisfies the conditions $(L.1)$, $(L.2)$, $(L.3)$, and by vacuously $(L.4)$ is fulfilled. So, Proposition 2.2 guarantees that there exists a map $\varphi_G : \mu^{-1}([c,d]) \times I \to \mu^{-1}([c,d])$ such that $\varphi_G(B,0) = B$, $\varphi_G(B,1) \in \mu^{-1}(d)$ for each $B \in \mu^{-1}([c,d])$, $\varphi_G^{-1}(A) = \{(A,0)\}$ and $\varphi_G(K,1) = K$ for each $K \in \mu^{-1}(d)$. Hence, the restriction map $\varphi_G|_{(\mu^{-1}([c,d])-\{A\})\times I} : (\mu^{-1}([c,d]) - \{A\}) \times I \to \mu^{-1}([c,d]) - \{A\}$ is a deformation retraction from $\mu^{-1}([c,d]) - \{A\}$ to $\mu^{-1}(d)$. In either case, apply Proposition 2.1 to conclude that (2.12.1) is true.

The proof of (2.12.2) is made in two cases. First, assume that $c = 0$. Apply $(F.1)$ and $(F.3)$ to get the inclusion $F^{-1}(A) \subseteq \{K \in \mathcal{C}(X) : A \subseteq K\} \times I$. From conditions $(F.1)$, $(F.2)$, $(F.3)$ and $(F.4)$, it follows that the restriction map $F|_{(\mu^{-1}([c,d])-\{A\})\times I} : (\mu^{-1}([c,d]) - \{A\}) \times I \to \mu^{-1}([c,d]) - \{A\}$ is a deformation retraction from $\mu^{-1}([c,d]) - \{A\}$ to $\mu^{-1}(c)$. Now, suppose that $c > 0$ and set $u = 0$. Conditions $(F.1)$, $(F.2)$ and $(F.3)$ imply that F fulfills $(L.1)$, $(L.2)$ and $(L.4)$. Condition $(L.3)$ is satisfied by vacuously. Thus, Proposition 2.2 ensures that there exists a map $\varphi_F : \mu^{-1}([c,d]) \times I \to \mu^{-1}([c,d])$ such that $\varphi_F(B,0) = B$, $\varphi_F(B,1) \in \mu^{-1}(c)$ for each $B \in \mu^{-1}([c,d])$, $\varphi_F^{-1}(A) = \{(A,0)\}$ and $\varphi_F(K,1) = K$ for each $K \in \mu^{-1}(c)$. This implies that the restriction map $\varphi_F|_{(\mu^{-1}([c,d])-\{A\})\times I} : (\mu^{-1}([c,d]) - \{A\}) \times I \to \mu^{-1}([c,d]) - \{A\}$ is a deformation retraction from $\mu^{-1}([c,d]) - \{A\}$ to $\mu^{-1}(c)$. Finally, invoke Proposition 2.1 to show that (2.12.2) holds. □

kkk

3. MAIN THEOREMS

Throughout this section X will be a dendrite, μ will be a Whitney map for $\mathcal{C}(X)$ and $c, d \in I$ will be such that $c < d$.

Theorem 3.1. *Let $A \in \mu^{-1}((c,d))$. The following statement are equivalent:*

- $(hl.1)$ *A makes a hole in $\mu^{-1}([c,d])$*
- $(hl.2)$ *A is cut point of $\mu^{-1}(\mu(A))$*
- $(hl.3)$ *There exist disjoint non-empty open subsets U and V of X such that $X \setminus A = U \cup V$ and, either $B \subseteq A \cup U$ or $B \subseteq A \cup V$ for each $B \in \mu^{-1}(\mu(A))$.*

Proof. The equivalence between $(hl.2)$ and $(hl.3)$ is proved in [11, Theorem 2.1, p. 210]. Let us show that $(hl.1)$ and $(hl.2)$ are equivalent.

Set $t = \mu(A)$, $\mathcal{W} = \mu^{-1}([t,d]) - \{A\}$ and $\mathcal{Y} = \mu^{-1}([c,t]) - \{A\}$. The sets $\mathcal{W}$ and $\mathcal{Y}$ are connected closed subsets of $\mu^{-1}([c,d]) - \{A\}$, $\mu^{-1}([c,d]) - \{A\} = \mathcal{W} \cup \mathcal{Y}$, $\mu^{-1}(t) - \{A\} = \mathcal{W} \cap \mathcal{Y}$, and by Theorem 2.12, $\mathcal{W}$ and $\mathcal{Y}$ have property b).

Now, assume that A is not a cut point of $\mu^{-1}(t)$. This means $\mu^{-1}(t) - \{A\}$ is connected. Since $\mu^{-1}([c,d]) - \{A\}$ is the union of its connected closed subsets $\mathcal{W}$ and $\mathcal{Y}$ having property b) whose intersection is connected, by [2, Proposition 8, p. 2001], $\mu^{-1}([c,d]) - \{A\}$ has property b) and so, as a consequence of this and [10, Theorem 2, p. 69], $\mu^{-1}([c,d]) - \{A\}$ is unicoherent. Then A does not make a hole in $\mu^{-1}([c,d])$. This proves that $(hl.1)$ implies $(hl.2)$.

Finally, suppose that $(hl.2)$ holds. Then $\mu^{-1}([c,d]) - \{A\}$ is the union of its connected closed subsets $\mathcal{W}$ and $\mathcal{Y}$ whose intersection is not connected. Hence, $\mu^{-1}([c,d]) - \{A\}$ is not unicoherent and so, A makes a hole in $\mu^{-1}([c,d])$. This completes the proof. ☐

An arc J contained in a continuum Z is a *free arc* proving that $J - E(J)$ is an open subset of Z.

Theorem 3.2. *If A makes a hole in $\mu^{-1}([c,d])$, then A is a free arc in X such that $Int_X A = A - E(A)$ and $c < \mu(A) < d$.*

Proof. First, since $\mu^{-1}([c,d]) - \{A\}$ is not unicoherent, it must not have property b). So, from Theorem 2.12, it follows that $\mu(A) \in (c,d)$.

Now, set $l = \mu(A)$ and let $g : X \to A$ be defined by $g(x) = r(x, A)$. Observe that g is a map. On the other hand, in light of Theorem 3.1, there exist disjoint non-emtpy open subsets U and V of $X - A$ such that $X - A = U \cup V$ and either $B \subseteq U \cup A$ or $B \subseteq V \cup A$ for each $B \in \mu^{-1}(l)$. Let us prove that $A \subseteq ab$ for each $(a,b) \in U \times V$ to conclude that $g(U)$ and $g(V)$ are disjoint one-point sets such that $E(A) = g(U) \cup g(V)$, $\mathrm{Cl}_X U = U \cup g(U)$ and $\mathrm{Cl}_X V = V \cup g(V)$.

Let $(a,b) \in U \times V$ and set $N = A \cap ab$. The fact that X is hereditarily unicoherent guarantees that N is a subcontinuum of X contained in A. Thus, $\mu(N) \leq \mu(A) = l$. On the other hand, if there were would exist $K \in \mu^{-1}(l)$ such that $ab \subseteq K$, then $K \cap U$ and $K \cap V$ would be non-empty sets, contradicting the choice of U and V. In conclusion, $\mu(ab) > l$. Define $\alpha : I \to \mathcal{C}(X)$ by

$$\alpha(t) = \{\lambda(x,x,N,t) : x \in ab\}.$$

Then α is well defined and continuous, $\alpha(0) = N$ and $\alpha(1) = ab$. Hence, there exists $s \in I$ such that $\mu(\alpha(s)) = l$. Let us prove that $s = 0$. Seeking a contradiction, assume that $s > 0$. Notice that $r(a,N) = r(a,A)$ and so $r(a,N)a - \{r(a,N)\}$ is contained in U. Since $\mu(r(a,N)\lambda(a,a,N,s)) = s\mu(r(a,N)a) > 0$, the inclusion $\lambda(a,a,N,s) \in r(a,N)a - \{r(a,N)\}$ holds. Then $\lambda(a,a,N,s) \in U$. Similarly, $\lambda(b,b,N,s) \in V$. From this, it follows that $\alpha(s) \cap U \neq \emptyset$ and $\alpha(s) \cap V \neq \emptyset$, a contradiction to the choice of U and V. Therefore, $s = 0$ and $N = \alpha(s)$. Thus, N is a subcontinuum of X contained in A and $\mu(A) = \mu(N)$. This implies that $A = N$ and $A \subseteq ab$. In conclusion, A is an arc and $E(A) = \{r(a,A), r(b,A)\}$. Hence, due a and b are arbitrary, each one of the sets $g(U)$ and $g(V)$ consists of exactly one point, $g(U) \cap g(V) = \emptyset$ and $E(A) = g(U) \cup g(V)$. Finally, note that $g^{-1}(g(U)) = U \cup g(U)$ and $g^{-1}(g(V)) = V \cup g(V)$ are closed subsets of X containing U and V, respectively. Hence, $U \subsetneq \mathrm{Cl}_X U \subseteq U \cup g(U)$ and $V \subsetneq \mathrm{Cl}_X V \subseteq V \cup g(V)$. These inclusions and the condition $g(U)$ and $g(V)$ are one-point sets together imply that $\mathrm{Cl}_X U = U \cup g(U)$ and $\mathrm{Cl}_X V = V \cup g(V)$.

Finally, since the equalities $\mathrm{Int}_X A = X - \mathrm{Cl}_X(X - A) = X - \mathrm{Cl}_X(U \cup V) = X - ((\mathrm{Cl}_X U) \cup (\mathrm{Cl}_X V)) = X - (U \cup g(U) \cup V \cup g(V)) = X - ((X - A) \cup E(A)) = X - (X - (A - E(A))) = A - E(A)$ hold, the set $A - E(A)$ is an open subset of X. Therefore, A is a free arc such that $\mathrm{Int}_X A = A - E(A)$. This completes the proof. □

Theorem 3.3. *If A is a free arc such that $\mathrm{Int}_X A = A - E(A)$ and $c < \mu(A) < d$, then A makes a hole in $\mu^{-1}([c,d])$.*

Proof. Set $l = \mu(A)$ and set $E(A) = \{w, z\}$. In light of Theorem 3.1, it suffices to find non-empty disjoint open subsets U and V of X such that $X - A = U \cup V$ and either $B \subseteq A \cup U$ or $B \subseteq A \cup V$ for each $A \in \mu^{-1}(l)$.

First, define $g : X \to A$ by $g(x) = r(x, A)$ for each $x \in X$. Notice that g is a map. On other hand, from our assumptions on A, it follows that $\mathrm{ord}_X(x) = 2$ for each $x \in \mathrm{Int}_X A$, $\mathrm{ord}_X(w) \geq 2$ and $\mathrm{ord}_X(z) \geq 2$. Then $g^{-1}(x) = \{x\}$ for each $x \in \mathrm{Int}_X A$ and, $g^{-1}(w)$ and $g^{-1}(z)$ are non-degenerate closed subsets of X such that $X = g^{-1}(w) \cup g^{-1}(z) \cup A$, $g^{-1}(w) \cap A = \{w\}$, $g^{-1}(z) \cap A = \{z\}$ and $g^{-1}(w) \cap g^{-1}(z) = \emptyset$.

Now, set $U = X - (A \cup g^{-1}(w))$ and $V = X - (A \cup g^{-1}(z))$. Notice that U and V are non-empty open subsets of X, $U \cup V = X - ((A \cup g^{-1}(w)) \cap (A \cup g^{-1}(z))) = X - A$ and $U \cap V = X - ((A \cup g^{-1}(w)) \cup (A \cup g^{-1}(z))) = \emptyset$. It only remains to argue that if $B \in \mu^{-1}(l)$, then either $B \subseteq A \cup U$ or $B \subseteq A \cup V$.

Let $B \in \mu^{-1}(l) - \{A\}$. Assume that $B \cap U \neq \emptyset$. Then, $B \cap g^{-1}(z) - \{z\} \neq \emptyset$. To see that $B \subseteq A \cup U$, it suffices to prove that $B \cap V = \emptyset$, or equivalently, $B \subseteq A \cup g^{-1}(z)$. If $A \cap B = \emptyset$, then $B \subseteq g^{-1}(z) \cup g^{-1}(w)$ and since B is connected, B must be a subset of $g^{-1}(z)$. Assume that $A \cap B \neq \emptyset$. From the facts that $B \subseteq g^{-1}(z) \cup (A \cup g^{-1}(w))$ and $g^{-1}(z) \cap (A \cup g^{-1}(w)) = \{z\}$, it follows that $z \in B$. Due A is an arc, $z \in E(A) \cap B$ and $A \neq B$, the inclusion $B \subseteq X - \{w\}$ holds. So, if $B \cap g^{-1}(w)$ were non-empty, then w would be an element of B and the inclusion $A \subseteq B$ would hold, a contradiction. In conclusion, $B \cap g^{-1}(w) = \emptyset$ and hence $B \subseteq A \cup g^{-1}(z)$. The proof is finished. □

Our last result presents the desired characterization and its proof follows immediately from theorems 3.2 and 3.3.

Corollary 3.4. *Then A makes a hole in $\mu^{-1}([c, d])$ if and only if A is a free arc such that $Int_X A = A - E(A)$ and $c < \mu(A) < d$.*

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Conclusiones

El trabajo de investigacin nos permite concluir los siguientes resultados:
Sea X una dendrita, μ una función de Whitney para $C(X)$ y $c, d \in I$ con $c < d$.

Teorema 1. *Si A agujera a $\mu^{-1}([c, d])$, entonces A es un arco libre en X tal que $Int_X A = A - E(A)$ y $c < \mu(A) < d$.*

Teorema 2. *Si A es un arco libre en X tal que $Int_X A = A - E(A)$ y $c < \mu(A) < d$, entonces A agujera a $\mu^{-1}([c, d])$.*

Corolario 1. *Un elemento $A \in \mu^{-1}([c, d])$ agujera a $\mu^{-1}([c, d])$ si y sólo si A es un arco libre en X tal que $Int_X A = A - E(A)$ y $c < \mu(A) < d$.*

Esto nos permitirá tener mayor conocimiento sobre los bloques de Whitney, ya que los bloques de Whitney son un tema específico que en teora de continuos y sus hiperespacios está tomando importancia.

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