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Increasing strong size properties and strong size block properties

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ABSTRACT

Let X be a continuum. The n-fold hyperspace $C_n(X)$, $n \in \mathbb{N}$, is the family of all nonempty closed subsets of X with at most n components, topologized with the Hausdorff metric. Let μ be a strong size map for $C_n(X)$. A strong size level is the subset $\mu^{-1}(t)$, with $t \in [0, 1]$. A strong size block is the subset $\mu^{-1}([s, t])$, with $0 \leq s < r \leq 1$. A topological property $\mathcal{P}$ is said to be a increasing strong size property provided that if μ is a strong size map for $C_n(X)$ and $t_0 \in [0, 1)$, is such that $\mu^{-1}(t_0)$ has property $\mathcal{P}$, then so does $\mu^{-1}(t)$ for each $t \in (t_0, 1)$. In this paper we show that uniform pathwise connectedness, uniform continuum-chainability and local connectedness are increasing strong size properties, and we will show some strong size block properties.

1. Introduction

H. Hosokawa defines strong size maps on the n-fold hyperspace of a continuum in [7] as a generalization of Whitney maps for the hyperspace of subcontinua of a continuum and proves the existence of such maps (see [7, Theorem 2.2, p. 956]). He also proves that some topological properties are strong size properties. Topological properties that are strong size properties have been studied in [17] and [18]. In this work we are interested in increasing strong size properties, which are the natural generalization of increasing Whitney properties. There are many works about properties that are increasing Whitney properties (see [8], [9], [13], [21], [23]).

In this paper we prove that the properties of being pathwise connectedness, continuum-chainability and locally connected continuum are increasing strong size properties, and we also show that being a one-

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dimensional continuum is not increasing strong size properties. Additionally we show some strong size block properties.

This paper is organized in four sections. After Introduction and Preliminaries, in Section 3 we present our results on increasing strong size properties. Finally, in Section 4 we present our results on strong size block properties.

2. Preliminaries

We will represent the set of *positive integers* by $\mathbb{N}$, the set of the *real numbers* by $\mathbb{R}$, the interval $[0,1]$ by I and S^1 to denote the unit circle in the Euclidean plane $\mathbb{R}^2$. Let Z be a topological space and let $A \subset Z$, we indicate the *closure of* A respect to Z by $\mathrm{Cl}_Z(A)$ and the *interior* of A by $\mathrm{Int}_Z(A)$ (when there is no confusion we omit the subscript Z). Given a metric space (Z,d), for $x \in Z$ and $\varepsilon > 0$ we define $\mathcal{V}(\varepsilon, x) = \{y \in Z : d(x,y) < \varepsilon\}$. Also, $\mathrm{diam}(Z)$ will denote the *diameter* of a space Z.

A *continuum* is a nonempty compact, connected, metric space. A *subcontinuum* of a space Z is a continuum contained in Z. Let X be a continuum with metric d. Let $n \in \mathbb{N}$. We consider the following hyperspaces of X:

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$

$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$$

and

$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}.$$

The hyperspace $C_n(X)$ is called the *n-fold hyperspace* of X (thus, $C_1(X)$ is the classical hyperspace of all subcontinua of X and, as is customary, is denoted by $C(X)$ instead of $C_1(X)$). The hyperspace $F_n(X)$ denotes the *n-fold symmetric product* of a continuum X. These spaces are topologized with the Hausdorff metric H, defined as follows: $H(A,B) = \inf\{\varepsilon > 0 : A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\}$, where $N(\varepsilon, C) = \{x \in X : d(x,c) < \varepsilon \text{ for some } c \in C\} = \bigcup\{\mathcal{V}(\varepsilon, x) : x \in C\}$ (see [20]).

A *map* means a continuous function. Let $A, B \in 2^X$. An *order arc* from A to B is a map $\alpha : I \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$, and if $0 \leq t < s \leq 1$, then $\alpha(t) \subsetneq \alpha(s)$. If α is an order arc from A to B in 2^X such that $A \in C(X)$, then α is contained in $C(X)$ (see [20, Lemma 1.11, p. 64]). Further, by [4, Proposition 2.6, p. 62] if α is an order arc in 2^X and $\alpha(0) \in C_n(X)$, then $\alpha(t) \in C_n(X)$ for all $t \in I$. Thus, this guarantees the existence of order arcs in $C_n(X)$.

A *Whitney map* for $C_n(X)$ is a mapping $\sigma : C_n(X) \to I$ such that $\sigma(\{x\}) = 0$ for each $x \in X$, $\sigma(X) = 1$ and if $A \subset B$ and $B \neq A$, then $\sigma(A) < \sigma(B)$. A *strong size map* for $C_n(X)$ is a map $\mu : C_n(X) \to I$ such that $\mu(A) = 0$ for each $A \in F_n(X)$, $\mu(X) = 1$ and if $A \subset B$, $A \neq B$ and $B \notin F_n(X)$, then $\mu(A) < \mu(B)$ (see [7]). Strong size maps on the n-fold hyperspace of a continuum as a generalization of Whitney maps for the hyperspace of subcontinua of a continuum. A *strong size level* is a subset of the form $\mu^{-1}(t)$ for a strong size map μ and $t \in I$. A *strong size block* is a subset of the form $\mu^{-1}([s,r])$ for a strong size map μ and $0 \leq s < r \leq 1$. Strong size levels and strong size blocks are always continua [7, Theorem 2.10, p. 958].

A topological property $\mathcal{P}$ is called:

- *Strong size property* if whenever X has property $\mathcal{P}$, so does every strong size level.
- *Increasing Whitney property* provided that if $t_0 \in [0,1)$ and $\sigma^{-1}(t_0)$ has property $\mathcal{P}$, then $\sigma^{-1}(t)$ has property $\mathcal{P}$ for each $t \in (t_0, 1)$.

- *Increasing strong size property* provided that if $t_0 \in [0, 1)$ and $\mu^{-1}(t_0)$ has property $\mathcal{P}$, then $\mu^{-1}(t)$ has property $\mathcal{P}$ for each $t \in (t_0, 1)$.
- *Strong size block property* if whenever X has property $\mathcal{P}$, so does every every strong size block.
- *Almost strong size block property* if whenever X has property $\mathcal{P}$, so does every every strong size block $\mu^{-1}([s, r])$, where $0 < s < r \leq 1$.

A continuum X is *irreducible* provided there exist $p, q \in X$ such that no proper subcontinuum of X contains both p and q.

A topological space Z is *unicoherent* provided that if $Z = A \cup B$, where A and B are closed connected subsets of Z, then $A \cap B$ is connected. We say that Z is *hereditarily unicoherent* provided that every subcontinuum of Z is unicoherent.

A continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua. We say that Z is *hereditarily decomposable* provided that every nondegenerate subcontinuum of Z is decomposable.

Let p, q be distinct points of X. Then X is said to be *aposyndetic at p with respect to q* if there is a subcontinuum K of X such that $p \in \mathrm{Int}(K)$ and $q \notin K$. If X is aposyndetic at p with respect to every point $q \in X$, then X is said to be *aposyndetic at p*. We say that X is *aposyndetic* if X is aposyndetic at every point of X. We say that X is *finitely aposyndetic*, if for any $p \in X$ and any finite subset F of X such that $p \notin F$, there exists a subcontinuum M of X such that $p \in \mathrm{Int}(M)$ and $M \cap F = \emptyset$. We say that X is *countable closed set aposyndetic*, if for any $p \in X$ and any countable closed subset F of X such that $p \notin F$, there exists a subcontinuum M of X such that $p \in \mathrm{Int}(M)$ and $M \cap F = \emptyset$.

A continuum X is said to be *uniformly pathwise connected* if there exists a family $\mathcal{F}$ of paths in X, satisfying:

1. For each $x, y \in X$ there exists a path $\alpha \in \mathcal{F}$ such that $\alpha(0) = x$ and $\alpha(1) = y$,
2. for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for each $\alpha \in \mathcal{F}$ there are numbers $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\mathrm{diam}(\alpha([t_{i-1}, t_i])) < \varepsilon$ for each $i \in \{0, 1, \ldots, k\}$.

A finite family of sets $\{A_1, \ldots A_k\}$ is said to be a *weak chain* provided that $A_i \cap A_j \neq \emptyset$ if $|i - j| \leq 1$.

A continuum X is said to be *continuum-chainable* provided that for each $\varepsilon > 0$ and for each pair of points $p, q \in X$, there exists a weak chain of continua $\{A_1, \ldots A_k\}$ such that $p \in A_1$, $q \in A_k$ and $\mathrm{diam}(A_i) < \varepsilon$ for each $i \in \{1, \ldots, k\}$.

A continuum X is said to be *uniformly continuum-chainable* provided that for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for each pair of points $p, q \in X$, there exists a weak chain of continua $\{A_1, \ldots A_k\}$ such that $p \in A_1$, $q \in A_k$ and $\mathrm{diam}(A_i) < \varepsilon$ for each $i \in \{1, \ldots, k\}$.

Let Z be a topological connected space. A map $f : Z \to S^1$ has a *lifting* if there exists a map $h : Z \to \mathbb{R}$ such that $f = \exp \circ h$, where exp is the map of $\mathbb{R}$ onto S^1 defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. A connected topological space Z has *property* (b) if each map $f : Z \to S^1$ has a lifting.

Let X be a continuum and let μ be a strong size map for $C_n(X)$. If $A \in C_n(X)$, let $\mathcal{O}(A) = \{B \in C_n(X) :$ each component of B intersects $A\}$ and let $\mathcal{O}^*(A) = \{B \in \mathcal{O}(A) : A \subset B\}$. If $t \in [0, \mu(A))$, let $C(A, t) = \{B \in \mu^{-1}(t) : B \subset A$ and each component of A intersects $B\}$.

The following two lemmas appear in [3].

Lemma 2.1. *([3, Lemma 3.1, p. 1143]). Let X be a continuum and let $\{A_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ be sequences of $C_n(X)$ such that $\lim A_k = A$, $\lim B_k = B$. If for each $k \in \mathbb{N}$, $B_k \in \mathcal{O}(A_k)$, then $B \in \mathcal{O}(A)$.*

Lemma 2.2. *([3, Lemma 3.2, p. 1143]). Let X be a continuum and let $\mu : C_n(X) \to I$ be a strong size map. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $A, B \in C_n(X)$ satisfy that $B \in \mathcal{O}(A)$, $A \subset N(\delta, B)$ and $|\mu(A) - \mu(B)| < \delta$, then $H(A, B) < \varepsilon$.*

In the lemma that follows we use of Unique Lifting Theorem, which asserts:

Theorem 2.3. *([5, Theorem 5.1, p. 21]). If Z is a connected space and $f, g : Z \to \mathbb{R}$ are two maps such that $\exp \circ f = \exp \circ g$ and $f(z) = g(z)$, for some $z \in Z$, then $f = g$.*

Lemma 2.4. *Let Z and Y be metric spaces having property* (b)*, let A be a connected subset of Y and let $f_Y : Y \to S^1$, $f_Z : Z \to S^1$ and $g : A \to Z$ be maps. If $h_Z : Z \to \mathbb{R}$ is a lifting of f_Z and $f_Y|_A = f_Z \circ g$, then there exists a lifting $h_Y : Y \to \mathbb{R}$ of f_Y such that $h_Y|_A = h_Z \circ g$.*

Proof. Let h_1 be a lifting of f_Y, *i.e.*, $f_Y = \exp \circ h_1$. Then $f_Y|_A = f_Z \circ g = \exp \circ h_1|_A$. Thus $\exp \circ h_Z \circ g = \exp \circ h_1|_A$. Let $y_0 \in A$, then there exists $k_0 \in \mathbb{Z}$ such that $(h_Z \circ g)(y_0) = h_1|_A(y_0) + k_0$. Let $h_Y(y) = h_1(y) + k_0$, for $y \in Y$. It is easy to see that h_Y is a lifting of f_Y. Even more, $\exp \circ h_Y|_A = \exp \circ (h_1|_A + k_0) = \exp \circ h_1|_A = \exp \circ h_Z \circ g$. By Theorem 2.3 we have that $h_Y|_A = h_Z \circ g$. □

Let $X_1, X_2, ..., X_m$ be a finite collection of subsets of X. We define the subset $< X_1, X_2, ..., X_m >_n$ of $C_n(X)$ by $< X_1, X_2, ..., X_m >_n = \{A \in C_n(X) : A \subset X_1 \cup ... \cup X_m$ and $A \cap X_k \neq \emptyset$ for $k = 1, ..., m\}$. Note that if $Y_1, Y_2, ..., Y_m$ are subsets of X and $X_i \subset Y_i$ for each $i = 1, 2, ..., m$, then $< X_1, X_2, ..., X_m >_n \subset < Y_1, Y_2, ..., Y_m >_n$. It can be demonstrated without much difficulty that if $X_1, X_2, ..., X_m$ are closed subsets of X, then $< X_1, X_2, ..., X_m >_n$ is closed in $C_n(X)$. It is known that the collection of all subsets of $C_n(X)$ of the form $< U_1, U_2, ..., U_m >_n$, where $U_1, U_2, ..., U_m$ are open subsets of X, is a base for the topology of $C_n(X)$ (see [15]).

3. Pathwise connectedness and continuum-chainability

In this section we will prove that pathwise connectedness and continuum-chainability are increasing strong size properties.

The second part of the next lemma is based on the idea in the proof of [11, Lemma 2, p. 2].

Lemma 3.1. *Let $\mu : C_n(X) \to I$ be a strong size map and let $t \in (0, 1)$. Then for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ satisfying that if $A, B \in \mathcal{O}^*(D) \cap \mu^{-1}(t)$ for some $D \in F_n(X)$, then there exists an arc $q : I \to \mu^{-1}(t)$ such that $q(0) = A$, $q(1) = B$ and $\operatorname{diam}(q([\frac{i-1}{m}, \frac{i}{m}])) < \varepsilon$ for each $i \in \{1, \ldots, m\}$.*

Proof. Let $D \in F_n(X)$ and let $A, B \in \mathcal{O}^*(D) \cap \mu^{-1}(t)$. By [17, Lemma 3.2, p. 106] there exists an arc $\gamma : I \to \mu^{-1}(t)$ such that $\gamma(0) = A$ and $\gamma(1) = B$. We will give the construction of γ just as the authors give it.

Let $\alpha', \beta' : I \to C_n(X)$ be two order arcs such that $\alpha'(0) = \beta'(0) = D$, $\alpha'(1) = A$ and $\beta'(1) = B$. Let α and β be parametrizations of these arcs by using the strong size map, i.e., $\alpha : [0, t] \to \alpha'(I)$ and $\beta : [0, t] \to \beta'(I)$, such that $\mu(\alpha(s)) = \mu(\beta(s)) = s$ for each $s \in [0, t]$.

Given $s \in [0, t]$. Define $f_s : [0, t] \to C_n(X)$ by $f_s(r) = \alpha(s) \cup \beta(r)$. Then f_s is well defined and is continuous. Since $\mu(f_s(0)) \leq t$ and $\mu(f_s(1)) \geq t$, there exists $r \in [0, t]$ such that $\mu(f_s(r)) = t$. Let $r_s = \sup\{r \in [0, t] : \mu(f_s(r)) = t\}$. So, we define $\gamma : [0, t] \to \mu^{-1}(t)$ by $\gamma(s) = \alpha(s) \cup \beta(r_s)$. Thus γ is well defined and continuous.

For the second part, let $\varepsilon > 0$. Let $\delta > 0$ be as in the Lemma 2.2 for the number ε. Let $\delta' > 0$ be as in the Lemma 2.2 for the number δ. Let $k \in \mathbb{N}$ such that $\frac{t}{k} < \delta'$. Take $i \in \{1, \ldots, k\}$. In order to prove that $\operatorname{diam}(\gamma([\frac{t(i-1)}{k}, \frac{it}{k}])) < \varepsilon$, let $v, w \in [\frac{t(i-1)}{k}, \frac{it}{k}]$ with $v < w$. Then $\alpha(v) \subset \alpha(w)$ and $\mu(\alpha(w)) - \mu(\alpha(v)) = w - v < \delta'$. By the choice of δ', $H(\alpha(v), \alpha(w)) < \delta$.

We need to prove that $r_w \leq r_v$. Suppose there exists $l \in \{r \in [0, t] : \mu(f_w(r)) = t\}$ such that $r_v < l$. Since $\beta(r_v) \subset \beta(l)$ and $\alpha(v) \subset \alpha(w)$, $f_v(r_v) \subset f_v(l) \subset f_w(l)$. Using $\mu(f_v(r_v)) = \mu(f_w(l)) = t$, we have $f_v(r_v) = f_v(l)$ and $\mu(f_v(l)) = t$, a contradiction. Thus $r_w \leq r_v$.

Since $\beta(r_w) \subset \beta(r_v)$ and $\alpha(w) \subset N(\delta, \alpha(v))$, $f_w(r_w) \subset N(\delta, f_v(r_v))$, which means that $\gamma(w) \subset N(\delta, \gamma(v))$. By the choice of δ, $H(\gamma(v), \gamma(w)) < \varepsilon$.

Therefore, the mapping $q : I \to \mu^{-1}(t)$ defined by $q(l) = \gamma(lt)$ has the required properties. $\square$

Recall the following. Let $u : C(C_n(X)) \to C_n(X)$ be a function given by $u(\mathcal{A}) = \bigcup\{A : A \in \mathcal{A}\}$. By [12, p. 23], u is a map, by [16, Lemma 7.2, p. 250] it is well defined, and it is clear that is onto. The map u is called the *union map*.

The following theorem is similar to a result for Whitney levels (see [21, Theorem 3.1, p. 191]).

Theorem 3.2. *Let X be a continuum, let $\mu : C_n(X) \to I$ be a strong size map and let $t_0 \in [0, 1)$. If $\mu^{-1}(t_0)$ is uniformly continuum-chainable, then $\mu^{-1}(t)$ is uniformly pathwise connected for each $t \in (t_0, 1)$.*

Proof. Let $t \in (t_0, 1)$ and let $\varepsilon > 0$. We will prove that $\mu^{-1}(t)$ is uniformly pathwise connected. Since the composition of μ and the union map u is uniformly continuous, there exists $\delta > 0$ such that if $\mathcal{A} \in C(\mu^{-1}(t_0))$ and $\operatorname{diam}(\mathcal{A}) < \delta$, then $\mu^{-1}(u(\mathcal{A})) < t$.

Let k be as in the definition of the uniform continuum-chainability of $\mu^{-1}(t_0)$ for the number δ. Let $A, B \in \mu^{-1}(t)$. Using order arcs, we can find A^* y $B^* \in \mu^{-1}(t_0)$ such that $A \in \mathcal{O}^*(A^*)$ and $B \in \mathcal{O}^*(B^*)$. Then there exists $\{\mathcal{A}_1, \dots, \mathcal{A}_k\}$ a weak chain of subcontinua of $\mu^{-1}(t_0)$ such that $A^* \in \mathcal{A}_1$ and $B^* \in \mathcal{A}_k$ and $\operatorname{diam}(\mathcal{A}_i) < \delta$ for each $i \in \{1, \dots, k\}$. Notice that $\mu(u(\mathcal{A}_i)) < t$ for each $i \in \{1, \dots, k\}$. Using order arcs, we can find an element $A_i \in \mu^{-1}(t)$ such that $u(\mathcal{A}_i) \subset A_i$. Clearly $\{A, A_1, \dots A_k, B\}$ is a weak chain.

Applying the Lemma 3.1 to each one of the pairs $\{A, A_1\}$, $\{A_1, A_2\}$, $\dots$, $\{A_k, B\}$, we find a path $\alpha : I \to \mu^{-1}(t)$ between A and B and a sequence of numbers $0 = r_0 < r_1, \dots, < r_{km} = 1$ such that $\operatorname{diam}[\alpha(r_{i-1}, r_i)] < \varepsilon$. $\square$

Note that the proof of [11, Theorem 2, p. 170] shows that a continuum uniformly pathwise connected is a continuum uniformly continuum-chainable. From this fact and Theorem 3.2 the following result is immediate.

Theorem 3.3. *Uniform pathwise connectedness and uniform continuum-chainability are increasing strong size properties.*

The method used in the proof of Theorem 3.2 can be used to prove the following theorem.

Theorem 3.4. *Let X be a continuum, let $\mu : C_n(X) \to I$ be a strong size map and $t_0 \in [0, 1)$. If $\mu^{-1}(t_0)$ is a chainable-continuum, then $\mu^{-1}(t)$ is pathwise connected for each $t \in (t_0, 1)$.*

Since pathwise connectedness implies continuum-chainability, the following corollary is a consequence of Theorem 3.4.

Corollary 3.5. *Pathwise connectedness and continuum-chainability are increasing strong size properties.*

It is known that pathwise connectedness is equivalent to arcwise connectedness in continua. Thus we have the following result.

Corollary 3.6. *Arcwise connectedness is an increasing strong size property.*

Corollary 3.7. *([7, Theorem 3.3, p. 963]). Arcwise connectedness is a strong size property.*

4. Strong size block properties

The main objective of this section is to prove that the following properties are strong size block properties: arcwise connectedness, locally connectedness and aposyndesis. The properties of being countable closed set aposyndetic and being finitely aposyndetic are almost strong size block properties.

Let $A \in C_n(X)$, $\mathcal{K}_A$ will denote the set $\{B \in C_n(X) : A \subset B\}$. The statement in [1, Lemma 13, p. 2004] was shown for $C(X)$ and 2^X, in the following lemma we will prove that it is also true for $C_n(X)$.

Lemma 4.1. *Let $\mathcal{K}$ be a nonempty subset of $C_n(X)$. If $\mathcal{K}_A \subset \mathcal{K}$, for each $A \in \mathcal{K}$, then $\mathcal{K}$ has property* (b).

Proof. Let $A \in C_n(X)$ and let $g : \mathcal{K}_A \to S^1$ be a map. Suppose that α and β are order arcs from A to X in $C_n(X)$. If h_α and h_β are liftings of $g|_\alpha$ and $g|_\beta$, respectively, such that $h_\alpha(X) = h_\beta(X)$, then $h_\alpha(A) = h_\beta(A)$. The proof of this fact is the same as [1, Lemma 12, p. 2004].

Finally, let $f : \mathcal{K} \to S^1$ be a map, we show that f has a lifting. Fix a number $t_0 \in \exp^{-1}(f(X))$. Let $A \in \mathcal{K}$ and let α_A be an order arc from A to X in $C_n(X)$. Since α_A is homeomorphic to I (see [20, Lemma 1.3, p. 57]), we have that α_A has property (b). Let $h_A : \alpha_A \to \mathbb{R}$ the unique map such that $f|_{\alpha_A} = \exp \circ h_A$ and $h_A(X) = t_0$. We define $h : \mathcal{K} \to \mathbb{R}$ by $h(A) = h_A(A)$. By the first part, h is well defined and, clearly, $f = \exp \circ h$. The proof of h being continuous is the same as the case for $C(X)$ and 2^X, as we had warned. Therefore, $\mathcal{K}$ has property (b). □

Theorem 4.2. *Let X be a continuum and let $\mu : C_n(X) \to I$ be a strong size map. If $t \in I$, then $\mu^{-1}([t, 1])$ has property* (b).

Proof. Notice that for each $A \in \mu^{-1}([t, 1])$, we have that $\{B \in \mu^{-1}([t, 1]) : A \subset B\} \subset \mu^{-1}([t, 1])$. Thus, by Lemma 4.1, $\mu^{-1}([t, 1])$ has property (b). □

Since property (b) implies unicoherence (see [24, Theorem 7.3, p. 227]), the following corollary is immediate.

Corollary 4.3. *Let X be a continuum and let $\mu : C_n(X) \to I$ be a strong size map. If $t \in I$, then $\mu^{-1}([t, 1])$ is unicoherent.*

In [2] the authors give a particular case of the following lemma. They assume that $\mathcal{H}$ is a subset of $C(X)$, where X has property (b). And they assume that $F_1(X) \subset \mathcal{H}$ and $C(A) \subset \mathcal{H}$, for all $A \in \mathcal{H}$, then their conclusion is that $\mathcal{H}$ has property (b). We present a generalization for $C_n(X)$, the proof is essentially the same.

Lemma 4.4. *Let $\mathcal{H}$ be a nonempty subset of $C_n(X)$, with $n \geq 3$. If $F_n(X) \subset \mathcal{H}$ and $C_n(A) \subset \mathcal{H}$, for all $A \in \mathcal{H}$, then $\mathcal{H}$ has property* (b).

Proof. Let $f : \mathcal{H} \to S^1$ be a map. We will demonstrate that f has a lifting, that is, there exists $h : \mathcal{H} \to \mathbb{R}$ such that $f = \exp \circ h$. By [14, Theorem 8, p. 177] $F_n(X)$ has property (b), thus there exists $h_1 : F_n(X) \to \mathbb{R}$ such that $f\big|_{F_n(X)} = \exp \circ h_1$.

On the other hand, by [16, Theorem 4.8, p. 244] $C_n(A)$ has property (b) for all $A \in \mathcal{H}$, thus $f\big|_{C_n(A)}$ has a lifting. Let $in : F_n(A) \to F_n(X)$ be the inclusion and let $\phi = f\big|_{C_n(A)}$, then $\phi\big|_{F_n(A)} = f\big|_{C_n(A)} \circ in$. By Lemma 2.4 there exists a lifting $h_A : C_n(A) \to \mathbb{R}$ of $f\big|_{C_n(A)}$ such that $h_A\big|_{F_n(A)} = h_1 \circ in$. Let $h : \mathcal{H} \to \mathbb{R}$ be the function defined by $h(A) = h_A(A)$.

Claim. h is a map. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim A_n = A_0$ in $\mathcal{H}$. Fix a sequence of elements $\{B_n\}_{n\in\mathbb{N}}$ in $F_n(X)$ converging to $B_0 \in F_n(X)$, such that $B_n \subset A_n$ and B_n intersect each component of

A_n, for each $n \in \mathbb{N} \cup \{0\}$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of order arcs in $C_n(X)$ such that $\cap\alpha_n = B_n$ and $\cup\alpha_n = A_n$. By the compactness of $2^{C_n(X)}$ there exists a subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ of $\{\alpha_n\}_{n\in\mathbb{N}}$, such that $\lim \alpha_{n_k} = \alpha_0$ and $\alpha_0 \in 2^{C_n(X)}$. Since $B_0, A_0 \in \alpha_0$, α_0 is non-degenerate. Then α_0 is an order arc in $C_n(X)$ (see [20, Theorem 1.4, p. 58]). Since $\cap\alpha_0 = B_0$ and $\cup\alpha_0 = A_0$, we have that α_0 is an order arc from B_0 to A_0 in $C_n(X)$ (see [20, Theorem 1.6, p. 59]).

Let $W = \{B_n : n \in \mathbb{N} \cup \{0\}\}$ and let $Y = (F_n(X) \times \{0\}) \cup (W \times I) \subset F_n(X) \times I$. It is to easy to see that $F_n(X) \times \{0\}$ is a deformation retract of Y. By [1, Proposition 9, p. 2001] Y has property (b). Let $\pi : Y \to \mathcal{H}$ defined by

$$\pi((B,t)) = \begin{cases} B & \text{if } t = 0 \\ \alpha_n(t) & \text{if } t > 0 \text{ and } B = B_n \end{cases}.$$

We have that $f \circ \pi : Y \to S^1$ has a lifting. Given that $F_n(X) \times \{0\}$ is a closed connected subset of Y and $\pi\left|_{F_n(X)\times\{0\}}\right.$ is a map in Y, then by Lemma 2.4, there exists $h_Y : Y \to \mathbb{R}$ a lifting of $f \circ \pi$ such that $h_Y((B,0)) = h_1 \circ \pi\left|_{F_n(X)\times\{0\}}\right. ((B,0))$ for each $B \in F_n(X)$.

To finish we will prove that $h_Y((B_n,1)) = h(A_n)$ for each $n \in \mathbb{N} \cup \{0\}$. For this:

$$\begin{aligned} \exp\circ(h_Y|_{B_n\times I}) &= (\exp\circ h_Y)|_{B_n\times I} = (f\circ\pi)|_{B_n\times I} \\ &= f\left|_{C_n(A_n)}\right. \circ \pi|_{B_n\times I} = (\exp\circ h_{A_n})\circ\pi|_{B_n\times I} \\ &= \exp\circ(h_{A_n}\circ\pi)|_{B_n\times I} \end{aligned}$$

and

$$\begin{aligned} (h_{A_n}\circ\pi\left|_{B_n\times I}\right.)((B_n,0)) &= h_{A_n}(\pi\left|_{B_n\times I}\right. ((B_n,0))) \\ &= h_{A_n}(B_n) = h_1(B_n) \\ &= h_Y((B,0)). \end{aligned}$$

By [1, Proposition 6, p. 2001] we have that $h_Y\left|_{B_n\times I}\right. = (h_{A_n}\circ\pi)\left|_{B_n\times I}\right.$, thus

$$\begin{aligned} h_Y((B_n,1)) &= (h_{A_n}\circ\pi\left|_{B_n\times I}\right.)((B_n,1)) \\ &= h_{A_n}(\pi\left|_{B_n\times I}\right. ((B_n,1))) = h_{A_n}(\alpha_n(1)) \\ &= h_{A_n}(A_n). \end{aligned}$$

Therefore $\lim h(A_n) = \lim h_Y((B_n,1)) = h_Y((B_0,1)) = h(A_0)$ and h is a map. It is clear that $f = \exp\circ h$. □

Theorem 4.5. *Let $n \geq 3$. Let X be a continuum and let $\mu : C_n(X) \to I$ be a strong size map. If $t \in I$, then $\mu^{-1}([0,t])$ has property* (b)*.*

Proof. Notice that $F_n(X) \subset \mu^{-1}([0,t])$ and $C_n(A) \subset \mu^{-1}([0,t])$ for each $A \in \mu^{-1}([0,t])$. Thus, by Lemma 4.4, $\mu^{-1}([0,t])$ has property (b). □

Corollary 4.6. *Let $n \geq 3$. Let X be a continuum and let $\mu : C_n(X) \to I$ be a strong size map. If $t \in I$, then $\mu^{-1}([0,t])$ is unicoherent.*

Theorem 4.7. *Let $n \geq 3$. Let $\mu : C_n(X) \to I$ be a strong size map and let $t \in I$. If X is a locally connected continuum, then $\mu^{-1}(t)$ has property* (b)*.*

Proof. By Theorem 4.5 $\mu^{-1}([0,t])$ has property (b). By [18, Theorem 6.9, p. 479] $\mu^{-1}(t)$ is a strong deformation retract of $\mu^{-1}([0,t])$. Finally, by [1, Proposition 9, p. 2001] $\mu^{-1}(t)$ has property (b). □

Corollary 4.8. *Let $n \geq 3$. Let $\mu : C_n(X) \to I$ be a strong size map and let $t \in I$. If X is a locally connected continuum, then $\mu^{-1}(t)$ is unicoherent.*

Theorem 4.9. *Let* $n \geq 3$*. Let* $\mu : C_n(X) \to I$ *be a strong size map and let* $0 \leq t_0 < t \leq 1$*. If* X *is a locally connected continuum, then* $\mu^{-1}([t_0, t])$ *has property* (b).

Proof. Following the proof of [18, Theorem 6.9, p. 479] we can demonstrate that $\mu^{-1}(t)$ is a strong deformation retract of $\mu^{-1}([t_0, t])$ for any $t_0 \in [0, t]$. By Theorem 4.7 and [1, Proposition 9, p. 2001] $\mu^{-1}([t_0, t])$ has property (b). □

Corollary 4.10. *Let* $n \geq 3$*. Let* $\mu : C_n(X) \to I$ *be a strong size map and let* $0 \leq t_0 < t \leq 1$*. If* X *is a locally connected continuum, then* $\mu^{-1}([t_0, t])$ *is unicoherent.*

Remark 4.11. It can easily be proved the follow affirmation. Let X be a continuum and let $\mu : C_n(X) \to I$ be a strong size map. Let $0 \leq t < s \leq 1$. If exists $t_0 \in [t, s]$ such that $\mu^{-1}(t_0)$ is arcwise connected, then $\mu^{-1}([t, s])$ is arcwise connected.

It is known from [7, Theorem 3.3, p. 963] that the property of being an arcwise connected continuum is a strong size property, so we have the following corollary which proves that being arcwise connected is a strong size block property.

Corollary 4.12. *Let* X *be a continuum arcwise connected. Let* $\mu : C_n(X) \to I$ *be a strong size map. If* $0 \leq t < s \leq 1$*, then* $\mu^{-1}([t, s])$ *is arcwise connected.*

Lemma 4.13. *Let* X *be a hereditarily decomposable and hereditarily unicoherent continuum, and let* $t \in (0, 1)$*. If* $\mu : C_n(X) \to I$ *be a strong size map, then* $\mu^{-1}(t)$ *contains a* $(2n-1)$*-cell.*

Proof. Let $t \in (0, 1)$. Since $\mu^{-1}(t) \cap (C_n(X) - C_{n-1}(X)) \neq \emptyset$ there are $B_1, \ldots, B_n$ pairwise disjoint subcontinua of X such that $\mu(B_1 \cup \cdots \cup B_n) = t$. It is easy to find $K_1, \ldots, K_n$ pairwise disjoint non-degenerate subcontinua of X such that $B_i \subset K_i$, for $i = 1, \ldots, n$. Thus $\mu(B_1 \cup \cdots \cup B_n) < \mu(K_1 \cup \cdots \cup K_n)$. Let α_i be order arc such that $\alpha_i(0) = \{k_i\}$ and $\alpha_i(1) = K_i$, where $k_i \in K_i$, for $i = 1, \ldots, n$. Let $s_1, \ldots, s_n \in I$ such that $\mu(\alpha_1(s_1) \cup \cdots \cup \alpha_n(s_n)) = t$. It is clear that $\alpha_1(s_1), \ldots, \alpha_n(s_n)$ are pairwise disjoint non-degenerate subcontinua of X.

Let $A = A_1 \cup \cdots \cup A_n$, where $A_i = \alpha_i(s_i)$ for $i = 1, \ldots, n$. For each $i \in \{1, 2, \ldots, n\}$, A_i is a decomposable and unicoherent subcontinuum of X. Suppose that $A_i = E_i \cup F_i$, where E_i and F_i are proper subcontinua of A_i. Let $\alpha_i : I \to C(X)$ and $\beta_i : I \to C(X)$ be order arcs such that $\alpha_i(0) = \beta_i(0) = E_i \cap F_i$, $\alpha_i(\frac{1}{2}) = E_i$, $\beta_i(\frac{1}{2}) = F_i$ and $\alpha_i(1) = \beta_i(1) = X$. Define the function $\xi : [0, 1]^{2n} \to C_n(X)$ by $\xi(t_1, \ldots, t_{2n}) = \bigcup_{i=1}^{n} \alpha_i(t_{2i-1}) \cup \beta_i(t_{2i})$. Then ξ is well defined and continuous.

The proof used in [18, Theorem 3.2, p. 470] can be used in order to prove that $\mu^{-1}(t)$ contains a $(2n-1)$-cell. □

An example with the properties from the next proposition is already known (see [22, Example 5.1, p. 1826]). We present another example.

Proposition 4.14. *The arc is an irreducible continuum such that none of its strong size levels is irreducible.*

Proof. It is known that $F_n(I)$ contains a n-cell. By Corollary 3.7, $\mu^{-1}(t)$ is an arcwise continuum. Since I is a hereditarily decomposable and hereditarily unicoherent continuum, by Lemma 4.13, $\mu^{-1}(t)$ contains a $(2n-1)$-cell for each $t \in (0, 1)$. Thus $\mu^{-1}(t)$ is not irreducible for each $t \in I$. □

The conclusion of next corollary is the same that [22, Theorem 5.3, p. 1827].

Corollary 4.15. *Irreducibility is not a strong size property.*

We do not know if there are continua such that its strong size levels are irreducible. In this way we perform the following question.

Question 4.16. Does there exist a continuum with irreducible strong size levels?

Proposition 4.17. *The property of being a one-dimensional space is not a strong size property, for* $n \geq 3$.

Proof. By [18, Corollary 3.3, p. 471] each strong size level contains a $(n-1)$-cell. $\square$

Lemma 4.18. *Let* X *be a continuum, let* $\mu : C_n(X) \to I$ *be a strong size map and let* $0 \leq t_0 < t$. *If* $\mathcal{K}$ *is a subcontinuum of* $C_n(X)$, $D \in \mu^{-1}(t)$ *and* $C(D, t_0) \subset \mathcal{K}$, *then* $D \in C(u(\mathcal{K}), t)$.

Proof. Notice that $\bigcup C(D, t_0) = D$. Thus $D \subset u(\mathcal{K})$. Let $A \in C(D, t_0)$. Since $A \in \mathcal{K}$, by [6, Lemma 3.1, p. 241] each component of $u(\mathcal{K})$ intersects A. Let K_C be a component of $u(\mathcal{K})$, since $A \subset D$, $K_C \cap D \neq \emptyset$. So, $D \in C(u(\mathcal{K}), t)$. $\square$

Lemma 4.19. *Let* X *be a continuum, let* $\mu : C_n(X) \to I$ *be a strong size map and* $0 \leq t_0 < s \leq r \leq t$. *If* $\mathcal{K}$ *is an open subset of* $\mu^{-1}(t_0)$, *then* $\mathcal{G} = \{D \in \mu^{-1}(r) : C(D, t_0) \subset \mathcal{K}\}$ *is an open subset of* $\mu^{-1}([s, t])$.

Proof. We will prove that $\mu^{-1}([s, t]) \setminus \mathcal{G}$ is a closed subset of $\mu^{-1}([s, t])$. We can assume that $\mu^{-1}([s, t]) \neq \mathcal{G}$ and $\mathcal{G} \neq \emptyset$. Let $\{B_k\}_{k \in \mathbb{N}}$ be a sequence of $\mu^{-1}([s, t]) \setminus \mathcal{G}$ that converges for some $B \in \mu^{-1}([s, t])$. Then, for each $k \in \mathbb{N}$, $C(B_k, t_0) \cap \mu^{-1}(t_0) \setminus \mathcal{K} \neq \emptyset$. By [7, Theorem 2.14, p. 960], for each $k \in \mathbb{N}$, $C(B_k, t_0) \in C(\mu^{-1}(t_0))$.

By compactness we may assume that the sequence $\{C(B_k, t_0)\}_{k \in \mathbb{N}}$ converges to an element $\mathcal{D} \in C(\mu^{-1}(t_0))$. Since $C(B_k, t_0) \cap \mu^{-1}(t_0) \setminus \mathcal{K} \neq \emptyset$, for each k, and $\mu^{-1}(t_0) \setminus \mathcal{K}$ is closed, we conclude that $\mathcal{D} \cap \mu^{-1}(t_0) \setminus \mathcal{K} \neq \emptyset$. Now, let $D \in \mathcal{D}$. Since $\lim C(B_k, t_0) = \mathcal{D}$, there exists $D_k \in C(B_k, t_0)$ for each $k \in \mathbb{N}$ such that $\lim D_k = D$. It follows from Lemma 2.1 that $D \in C(B, t_0)$. Then $C(B, t_0) \cap \mu^{-1}(t_0) \setminus \mathcal{K} \neq \emptyset$ and $B \in \mu^{-1}([s, t]) \setminus \mathcal{G}$. Therefore $\mathcal{G}$ is an open subset of $\mu^{-1}([s, t])$. $\square$

Theorem 4.20. *Let* X *be a continuum, let* $\mu : C_n(X) \to I$ *be a strong size map and* $0 \leq t_0 \leq s < t \leq 1$. *If* $\mu^{-1}(t_0)$ *is a locally connected continuum, then* $\mu^{-1}([s, t])$ *is a locally connected continuum.*

Proof. Let $A = A_1 \cup \cdots \cup A_s \in \mu^{-1}([s, t])$, where A_i, $i \in \{1, \ldots, s\}$ are the components of A. Let $\varepsilon > 0$ such that $\mathcal{V}(\varepsilon, A)$ is a neighborhood of A in $\mu^{-1}([s, t])$. Let $\delta > 0$ be as in the Lemma 2.2 for the number ε. Fix $0 < \gamma < \frac{\delta}{2}$ such that $\mathrm{Cl}(N(\gamma, A_i)) \cap \mathrm{Cl}(N(\gamma, A_j)) = \emptyset$ if $i \neq j$.

Let $\mathcal{U} = \{D \in \mu^{-1}(t_0) : D \subset N(\frac{\gamma}{2}, A)\}$. Notice that $\mathcal{U}$ is an open subset of $\mu^{-1}(t_0)$ and $C(A, t_0) \subset \mathcal{U}$. Let $\mathcal{K}$ be a component of $\mathcal{U}$ such that $C(A, t_0) \subset \mathcal{K}$. Since $\mu^{-1}(t_0)$ is locally connected, $\mathcal{K}$ is open in $\mu^{-1}(t_0)$. Let $K = \bigcup \mathrm{cl}_{\mu^{-1}(t_0)}(\mathcal{K})$. By [15, Lemma 3.1, p. 258], $K \in C_n(X)$ and by Lemma 4.18, $A \in C(K, \mu(A))$.

Let $x_i \in A_i$, for each $i \in \{2, \ldots, s\}$. By [19, Corollary 5.5, p. 74], there exists a subcontinuum L^* of X such that $A_1 \subset L^*$, $A_1 \neq L^*$ and $L^* \subset N(\gamma, A_1)$. Let $L = \bigcup_{i \neq 1} A_i \cup L$. Notice that $\mu(A) < \mu(L)$. Then $C(A, t_0) \subset C(L, t_0) \subset \mathcal{U}$ and $C(L, t_0) \neq C(A, t_0)$. Thus $C(L, t_0) \subset \mathcal{K}$. Let $E \in C(L, t_0) \setminus C(A, t_0)$. Then, $E \cap X \setminus A \neq \emptyset$. Since $E \subset K$, $K \cap X \setminus A \neq \emptyset$. Thus $K \neq A$.

Notice that $K \subset N(\delta, A)$. Now for each $F \in C(K, \mu(A))$, by Lemma 2.2, $H(A, F) < \varepsilon$. Hence $C(K, \mu(A)) \subset \mathcal{V}(\varepsilon, A)$. Let $\mathcal{G} = \{D \in \mu^{-1}(\mu(A)) : C(D, t_0) \subset \mathcal{K}\}$. Clearly $A \in \mathcal{G}$. By Lemma 4.19, $\mathcal{G}$ is an open subset of $\mu^{-1}([s, t])$. Let $D \in \mathcal{G}$. By Lemma 4.18, $D \in C(K, \mu(A))$. Then $\mathcal{G} \subset C(K, \mu(A))$.

Thus, since $C(K, \mu(A))$ is connected (see [7, Theorem 2.14, p. 959]) and $A \in \mathrm{Int}(C(K, \mu(A))) \subset C(K, \mu(A)) \subset \mathcal{V}(\epsilon, \mathcal{A})$, we have that $\mu^{-1}([s, t])$ is locally connected. $\square$

Corollary 4.21. *The property of being a locally connected continuum is an increasing strong size property.*

It is well known that the property of being a locally connected continuum is a strong size property (see [7, Theorem 3.1, p. 962]), so we have the following corollary which proves that being locally connected is a strong size block property.

Corollary 4.22. *Let X be a locally connected continuum and let $\mu : C_n(X) \to I$ be a strong size map. If $0 \leq t < s \leq 1$, then $\mu^{-1}([t,s])$ is locally connected.*

Lemma 4.23. *Let $K_1, K_2, ..., K_m$ be subcontinua of X and let $\mu : C_n(X) \to I$ be a strong size map. If $K_i \cap K_j = \emptyset$ for each distinct pair $i, j \in \{1, ...m\}$, then the set $< K_1, K_2, ..., K_m >_n \cap \mu^{-1}([s,t])$ is connected for $0 \leq s < t \leq 1$.*

Proof. Put $K = K_1 \cup ... \cup K_m$. Since each K_i is nonempty, the definition of $< K_1, K_2, ..., K_m >_n$ implies $K \in< K_1, K_2, ..., K_m >_n$. Let $A, B \in< K_1, K_2, ..., K_m >_n \cap \mu^{-1}([s,t])$, such that $A \neq B$. Suppose that $A \neq K$ and $B \neq K$. By the [20, Theorem 1.8, p. 59] there exist order arcs α_1 and α_2 from A to K and from B to K, respectively, contained in $< K_1, K_2, ..., K_m >_n$. If $\mu(K) \in [s,t]$, then we have $< K_1, K_2, ..., K_m >_n \cap \mu^{-1}([s,t])$ is arcwise connected. Suppose $\mu(K) > t$ and consider $t_1, t_2 \in [s,t]$ such that $\mu(\alpha_1(t_1)) = \mu(\alpha_2(t_2)) = t$. As $< K_1, K_2, ..., K_m >_n \cap \mu^{-1}(t)$ is arcwise connected (see the Proof of [7, Theorem 2.14, p. 960]), there exists an arc from $\mu(\alpha_1(t_1))$ to $\mu(\alpha_2(t_2))$. Thus there exists an arc from A to B. So $< K_1, K_2, ..., K_m >_n \cap \mu^{-1}([s,t])$ is connected. □

The next theorem proves that being aposyndetic is a strong size block property.

Theorem 4.24. *Let $\mu : C_n(X) \to I$ be a strong size map and $0 \leq s < t \leq 1$. If X is an aposyndetic continuum, then $\mu^{-1}([s,t])$ is aposyndetic.*

Proof. Let $A, B \in \mu^{-1}([s,t])$ such that $A \neq B$. Note that $A \setminus B \neq \emptyset$. Let $a \in A \setminus B$. Since X is aposyndetic, for each $x \in B$, there exists K_x subcontinuum such that $x \in int(K_x)$ and $a \notin K_x$. By compactness of B, there exists a subset $\{x_1, ..., x_r\} \subset B$ such that $B \subset \cup_{j=1}^{r} int(K_{x_j})$ and $a \notin \cup_{j=1}^{r} K_{x_j}$.

If B is connected, then we consider $K = \cup_{j=1}^{r} K_{x_j}$. By the Lemma 4.23 we have that $\langle K \rangle_n \cap \mu^{-1}([s,t])$ is a continuum neighborhood of B in $\mu^{-1}([s,t])$ which does not contain A.

If B is not connected, then we can assume without loss of generality that $K_{x_1}, ..., K_{x_r}$ are pairwise disjoint. By the Lemma 4.23 we have that $\langle K_{x_1}, ..., K_{x_r} \rangle_n \cap \mu^{-1}([s,t])$ is a continuum neighborhood of B in $\mu^{-1}([s,t])$ which does not contain A.

In any case, $\mu^{-1}([s,t])$ is aposyndetic at B with respect to A. Therefore $\mu^{-1}([s,t])$ is aposyndetic. □

Lemma 4.25. *Let $A \in \mu^{-1}([s,t])$ with $0 < s < t \leq 1$. Let $\mathcal{C}$ be a closed subset of $\mu^{-1}([s,t])$ such that $A \notin \mathcal{C}$ and for all $B \in \mathcal{C}$ we have $B \nsubseteq A$ or $A \nsubseteq B$. Then there exists $\varepsilon > 0$ such that for all $B \in \mathcal{C}$ we have $B \nsubseteq \mathcal{V}(\varepsilon, A)$.*

Proof. Suppose the statement is false. Then for all $m \in \mathbb{N}$ there exists $B_m \in \mathcal{C}$ such that $B_m \subset \mathcal{V}(1/m, A)$. Since $\mathcal{C}$ is a closed subset of $\mu^{-1}([s,t])$, it is closed in $C_n(X)$. Thus, there exists $B' \in \mathcal{C}$ such that $\lim B_m = B'$. And therefore $B' \subset A$, a contradiction. □

Lemma 4.26. *Let $A \in C_n(X)$ such that $\mu(A) \in (0,1)$. Let $\mathcal{C}$ be a closed subset of $C_n(X)$ such that for all $B \in \mathcal{C}$ we have $B \subsetneq A$ or $A \subsetneq B$. Then there exist $s_0, t_0 \in I$ such that $s_0 < \mu(A) < t_0$ and $\mu^{-1}([s_0, t_0]) \cap \mathcal{C} = \emptyset$. If $A \subsetneq B$ for all $B \in \mathcal{C}$, then there exists $t_0 \in I$ such that $\mu(A) < t_0$ and $\mu^{-1}([\mu(A), t_0]) \cap \mathcal{C} = \emptyset$. And, if $B \subsetneq A$ for all $B \in \mathcal{C}$, then there exists $s_0 \in I$ such that $s_0 < \mu(A)$ and $\mu^{-1}([s_0, \mu(A)]) \cap \mathcal{C} = \emptyset$.*

Proof. If $\mathcal{C}$ is a closed subset of $C_n(X)$, then $\mathcal{C}$ is compact. Thus $\mu(\mathcal{C})$ is a closed subset of I such that $\mu(A) \notin \mu(\mathcal{C})$. □

The next theorem proves that being countable closed set aposyndetic is an almost strong size block property.

Theorem 4.27. *Let* $\mu : C_n(X) \to I$ *be a strong size map and let* $0 < s < t \leq 1$. *If* X *is countable closed set aposyndetic, then* $\mu^{-1}([s,t])$ *is countable closed set aposyndetic.*

Proof. Let $\mathcal{A}$ be a countable closed subset of $\mu^{-1}([s,t])$. So $\mu^{-1}([s,t]) \setminus \mathcal{A} \neq \emptyset$. Let $B \in \mu^{-1}([s,t]) \setminus \mathcal{A}$. In order to prove the theorem we will consider three cases.

Case 1) Suppose $s < \mu(B) < t$. Let $\mathcal{A}_0 \subset \mathcal{A}$ such that $B \subsetneq C$ or $C \subsetneq B$, for all $C \in \mathcal{A}_0$. By Lemma 4.26 there exist $s_0, t_0 \in I$ such that $s < s_0 < \mu(B) < t_0 < t$ and $\mu^{-1}([s_0,t_0]) \cap \mathcal{A}_0 = \varnothing$. Let $\mathcal{A}_1 = \mathcal{A} \cap \mu^{-1}([s_0,t_0])$. Clearly $\mathcal{A}_1$ is a countable closed subset of $\mu^{-1}([s_0,t_0])$. By the Lemma 4.25 there exists $\varepsilon > 0$ such that for all $A \in \mathcal{A}_1$ we have $A \nsubseteq \mathcal{V}(\varepsilon, B)$. If $U = X \setminus \mathrm{Cl}_X(\mathcal{V}(\epsilon/2, B))$, then by [10, Theorem 21, p. 58] there exists a map $S : \mathcal{A}_1 \to X$ such that $S(A) \in A \cap U$ for each $A \in \mathcal{A}_1$. Thus $S(\mathcal{A}_1)$ is a countable closed subset of X, such that $S(\mathcal{A}_1) \cap B = \emptyset$. Then, for each $x \in B$, there is a subcontinuum K_x such that $x \in \mathrm{Int}_X(K_x) \subset K_x \subset X \setminus S(\mathcal{A}_1)$. Since B is compact, there is a subset $\{x_1, ..., x_r\} \subset B$ such that $B \subset \cup_{j=1}^{r} \mathrm{Int}(K_{x_j})$. Without loss of generality, we can assume that $K_{x_1}, ..., K_{x_r}$ are pairwise disjoint. By Lemma 4.23 $\langle K_{x_1}, ..., K_{x_r} \rangle_n \cap \mu^{-1}([s_0,t_0])$ is a subcontinuum of $\mu^{-1}([s_0,t_0])$, such that $B \in \mathrm{Int}_{C_n(X)}(\langle K_{x_1}, ..., K_{x_r} \rangle_n) \cap \mu^{-1}([s_0,t_0])$. Since for each $j \in \{1, ..., r\}$, $S(\mathcal{A}_1) \cap K_{x_j} = \emptyset$, we have that $A \nsubseteq \cup_{j=1}^{r} K_{x_j}$ for all $A \in \mathcal{A}_1$. Thus $(\langle K_{x_1}, ..., K_{x_r} \rangle_n \cap \mu^{-1}([s_0,t_0])) \cap \mathcal{A}_1 = \emptyset$ and $(\langle K_{x_1}, ..., K_{x_r} \rangle_n \cap \mu^{-1}([s_0,t_0])) \cap \mathcal{A} = \emptyset$.

Case 2) Suppose that $\mu(B) = s$. Let $\mathcal{A}_0 \subset \mathcal{A}$ such that $B \subsetneq C$, for all $C \in \mathcal{A}_0$. By Lemma 4.26 there exists $t_0 \in I$ such that $s < t_0 < t$ and $\mu^{-1}([s,t_0]) \cap \mathcal{A}_0 = \emptyset$. Let $\mathcal{A}_1 = \mathcal{A} \cap \mu^{-1}([s,t_0])$. Proceeding in the same way as Case 1) we get pairwise disjoint subcontinua $K_{x_1}, ..., K_{x_r}$ such that $(\langle K_{x_1}, ..., K_{x_r} \rangle_n \cap \mu^{-1}([s_0,t_0])) \cap \mathcal{A}_1 = \emptyset$ and $(\langle K_{x_1}, ..., K_{x_r} \rangle_n \cap \mu^{-1}([s_0,t_0])) \cap \mathcal{A} = \emptyset$.

Case 3) Suppose that $\mu(B) = t$. Let $\mathcal{A}_0 \subset \mathcal{A}$ such that $C \subsetneq B$, for all $C \in \mathcal{A}_0$. By Lemma 4.26 there exists $s_0 \in I$ such that $s < s_0 < t$ and $\mu^{-1}([s_0,t]) \cap \mathcal{A}_0 = \emptyset$. Let $\mathcal{A}_1 = \mathcal{A} \cap \mu^{-1}([s_0,t])$. Proceeding in the same way as Case 1) we get pairwise disjoint subcontinua $K_{x_1}, ..., K_{x_r}$ such that $(\langle K_{x_1}, ..., K_{x_r} \rangle_n \cap \mu^{-1}([s_0,t_0])) \cap \mathcal{A}_1 = \emptyset$ and $(\langle K_{x_1}, ..., K_{x_r} \rangle_n \cap \mu^{-1}([s_0,t_0])) \cap \mathcal{A} = \emptyset$. □

Corollary 4.28. *Let* $\mu : C_n(X) \to I$ *be a strong size map and let* $0 < s < t \leq 1$. *If* X *is finitely aposyndetic, then* $\mu^{-1}([s,t])$ *is finitely aposyndetic.*

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