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ORIGINAL PAPER

The Hyperspace $F_n^K(X)$

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Abstract

Given a metric continuum X and a positive integer n, $F_n(X)$ denotes the hyperspace of all nonempty subsets of X with at most n points endowed with the Hausdorff metric. For any $K \in F_n(X)$, we define $F_n(K, X)$ as the collection of elements of $F_n(X)$ containing K and we consider $F_n^K(X)$ as the quotient space obtained from $F_n(X)$ by shrinking $F_n(K, X)$ to one point set, endowed with the quotient topology. In this paper, we report the first results of the investigation related with this hyperspace. We focus our attention on proving results regarding to aposyndesis, local connectedness, arcwise connectedness, unicoherence, and cut points of $F_n^K(X)$.

Keywords Continuum · Hyperspaces · Quotient space · Symmetric products · Unicoherence · Aposyndesis

Mathematics Subject Classification 54B15 · 54B20 · 54F15

1 Introduction

Let X be a topological space, and by a *hyperspace* of X, we mean a specified collection of subsets of X endowed with the Vietoris topology. Some hyperspaces that have been studied and appear in the literature are (see [12] and [24]):

$$CL(X) = \{A \subset X : A \text{ is nonempty and closed in } X\};$$
$$2^X = \{A \in CL(X) : A \text{ is compact}\};$$
and given a positive integer n,
$$F_n(X) = \{A \subset X : A \text{ has at most } n \text{ points}\};$$
$$C_n(X) = \{A \in 2^X : \text{A has at most } n \text{ points and is nonempty}\}.$$

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If X is a T_1-space, $F_n(X)$ is called the n^{th}*-symmetric product* of X. A *continuum* is a nonempty compact connected metric space. Given a continuum X, a *subcontinuum* of X is a nonempty closed connected subset of X, $C_1(X)$ is called the *hyperspace of subcontinua of* X and it is denoted by $C(X)$. Symmetric products were introduced by K. Borsuk and S. Ulam in [2]. It is well known that if X is a continuum, then 2^X and $C(X)$ are continua (see [24, Theorem 1.13, p. 65]).

Other classes of hyperspaces that have been studied are the quotient spaces between hyperspaces; that is, if $\mathcal{H}(X)$ is a hyperspace for X and $\mathcal{F}$ is a specific collection of elements of $\mathcal{H}(X)$, then the quotient space $\mathcal{H}(X)/\mathcal{F}$ is obtained from $\mathcal{H}(X)$ by shrinking $\mathcal{F}$ to one point set. The quotient spaces between hyperspaces are a powerful tool to study the continuum for which they are defined and, of course, they provide new problems to solve. In the literature, we can find the following quotient spaces between hyperspaces:

In 1979, S. B. Nadler Jr. introduced the *hyperspace suspension of a continuum* X as the quotient space $C(X)/F_1(X)$; in particular, he showed that this hyperspace has the fixed point property for any chainable continuum (see [22]). In 2004, S. Macías defined the n*- fold hyperspace suspension of a continuum* X as the quotient space $C_n(X)/F_n(X)$; he showed results related to finitely aposyndetic and locally connectedness (see [19]). In 2010, F. Barragan defined the n*-fold symmetric product suspension of a continuum* X as the quotient space $F_n(X)/F_1(X)$ and he studied some properties of this space such as unicoherence, local connectedness and arcwise connectedness (see [1]). In 2013, E. Castañeda-Alvarado and J. Sánchez-Martínez generalized the n-fold symmetric product suspension of a continuum X, they introduced the quotient space $F_n(X)/F_m(X)$ with $m < n$, and in particular, they analyzed the unicoherence of this space (see [4]).

In order to introduce the hyperspace that we study in this paper, given a continuum X, $n > 1$, and $K \in F_n(X)$, we consider $F_n(K, X) = \{A \in F_n(X) : K \subset A\}$ of $F_n(X)$ and we define the hyperspace:

$$F_n^K(X) = F_n(X)/F_n(K, X)$$

which is obtained from $F_n(X)$ by shrinking $F_n(K, X)$ to one point set, endowed with the quotient topology.

From the definition, we have the following properties:

(1) If K has n points, then $F_n(K, X) = \{K\}$, and thus, $F_n^K(X)$ is homeomorphic to $F_n(X)$.
(2) If K has l points where $1 \leq l < n$, then $F_n(K, X)$ is the continuous image of $F_m(X)$ with $m = n-l$ (to know $h : F_m(X) \to F_n(K, X)$ given by $h(A) = A \cup K$), and therefore, $F_n(K, X)$ is also a continuum.

Since $F_n(K, X)$ is a continuum for each $n > 1$ and each $K \in F_n(X)$, by [23, Theorem 3.10, p. 40], we have that $F_n^K(X)$ is a continuum.
To study $F_n^K(X)$, this paper is organized as follows.

In Sect.2, we introduce general notation and definitions that we will use along the paper.

In Sect. 3, we give the geometric models for $F_2^K(X)$ when X is the unit interval and the unit circumference, which will be useful in future sections.

In Sect. 4, we study the connectedness of $F_n^K(X)$; in particular, we classify local connectedness and arcwise connectedness of $F_n^K(X)$, and also, we establish the relationship between cut points of X and $F_n^K(X)$.

In Sect. 5, we give conditions for K under which the hyperspace $F_n^K(X)$ is aposyndetic for each $n > 1$.

In Sect. 6, we provide a characterization of the unit circumference in terms of the space $F_2^{\{p\}}(X)$.

In Sect. 7, we prove that $F_n^K(X)$ is unicoherent for each continuum X, for each $n > 1$, and for each $K \in F_{n-1}(X)$.

In Sect. 8, we show that given a finite graph X, $F_2(X)$ is homeomorphic to $F_2^{\{p\}}(X)$ if and only if X is an arc or a simple n-od and p is an end point of X.

2 Preliminaries

The symbols $\mathbb{N}$ and $\mathbb{R}$ will denote the positive integers and the real numbers, respectively. If X is a topological space and $Y \subset X$, then $X - Y$ is considered as a subspace of X, also $\mathrm{Cl}_X(Y)$, $\mathrm{Bd}_X(Y)$ and $\mathrm{Int}_X(Y)$ denote the closure, the boundary, and the interior of Y in X, respectively. A *map* means a continuous function. An onto map $f : X \to Y$ between continua is said to be *monotone* provided that for any point $y \in Y$, $f^{-1}(y)$ is a connected subset of X. An *arc* is any space homeomorphic to the interval $I = [0, 1]$; a *simple closed curve* is any space homeomorphic to the unit circumference S^1 in the Euclidean plane $\mathbb{R}^2$.

Given a continuum X with metric d, $x \in X$, and $\epsilon > 0$, we define the *open ball of radius* ϵ *with center* x as $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$. In this paper, H denotes the Hausdorff metric on 2^X induced by d (see [24, Theorem 0.2, p. 02]). Given a finite collection, $U_1, U_2, ..., U_m$, of subsets of X, $\langle U_1, U_2, ..., U_m\rangle_n$ denotes the following subset of $F_n(X)$:

$$\left\{A \in F_n(X) : A \subset \bigcup_{i=1}^{m} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, ..., m\right\}.$$

It is known that the family of all subsets of $F_n(X)$ of the form $\langle U_1, U_2, ..., U_m\rangle_n$, where each U_i is an open subset of X, is a basis for a topology on $F_n(X)$ (see [24, Theorem 0.11, p. 09]), which is called the *Vietoris topology*. It is well known that the Vietoris topology and the topology induced by the Hausdorff metric coincide (see [24, Theorem 0.13, p. 10]).

A *finite graph* is a continuum which is a finite union of arcs, such that each two of them meet at a finite set. Given a finite graph X, $x \in X$ and $n \in \mathbb{N}$, we say that the *order* of x in X is n (denoted by $\mathrm{ord}(x, X) = n$), provided that for every $\epsilon > 0$, there exists an open neighborhood U of x in X with diameter less than ϵ, such that the boundary of U has exactly n points. If $x \in X$ and $\mathrm{ord}(x, X) = 1$, then x is called an *end point*, and if $\mathrm{ord}(x, X) \geq 3$, then x is called an *ramification point* of X. By a *simple n-od* ($n \geq 3$), we mean a finite graph X with only one ramification point, exactly n end points and without simple closed curves. A simple 3-od will be called a *simple triod*.

We denote by e the exponential map from the real line $\mathbb{R}$ to S^1. A map $f : X \to S^1$ is *inessential*, if there exists a map $h : X \to \mathbb{R}$, such that $f = e \circ h$; in other case, we say that f is *essential*.

A continuum X is said to be:

- *unicoherent* provided that whenever A and B are subcontinua of X, such that $X = A \cup B$, then $A \cap B$ is connected;
- *locally connected at a point* $x \in X$ provided that for each open neighborhood U of x in X, there exists a connected open neighborhood V of x in X, such that $x \in V \subset U$. X is *locally connected* if X is locally connected in each of its points;
- *connected im kleinen at* $x \in X$, written *cik* at x, provided that every neighborhood of x contains a connected neighborhood of x;
- *arcwise connected* provided that any two different points can be joined by an arc;
- *aposyndetic at* x *with respect to* y, provided that there exists a subcontinuum W of X, such that $x \in \mathrm{Int}_X(W) \subset W \subset X - \{y\}$, it is said to be *aposyndetic at* x, if it is aposyndetic at x with respect to any point of $X - \{x\}$, and it is said to be *aposyndetic,* if it is aposyndetic at each of its points.

Finally, given $x \in X$, x is called a *non-cut point* if $X - \{x\}$ is connected and it is called a *cut point* otherwise.

Given a continuum X, $n > 1$, and $K \in F_n(X)$, for the study of the quotient space $F_n^K(X)$, we will adopt the following notation.

$\pi_{n,K}^X : F_n(X) \to F_n^K(X)$ will denote the quotient map. If $A \in F_n(X) - F_n(K, X)$, $\pi_{n,K}^X(A)$ will be denoted by $[A]$, and if $A \in F_n(K, X)$, then $\pi_{n,K}^X(A)$ will be denoted by $F_{n,K}^X$. With this notation, we have:

$$\pi_{n,K}^X(A) = \begin{cases} [A] & \text{if } A \in F_n(X) - F_n(K, X), \\ F_{n,K}^X & \text{if } \quad A \in F_n(K, X). \end{cases}$$

Since $F_n(K, X)$ is a subcontinuum of $F_n(X)$ [see (1) and (2) of the Introduction], we have that $\pi_{n,K}^X$ is a monotone map.

If $K = \{p\}$, we will write $F_n^p(X)$, $F_n(p, X)$, $\pi_{n,p}^X$, and $F_{n,p}^X$ instead of $F_n^{\{p\}}(X)$, $F_n(\{p\}, X)$, $\pi_{n,\{p\}}^X$, and $F_{n,\{p\}}^X$, respectively.

Remark 2.1 Given a continuum X, $n > 1$ and $K \in F_n(X)$, it follows directly from the definitions that $F_n^K(X) - \{F_{n,K}^X\}$ is homeomorphic to $F_n(X) - F_n(K, X)$ and $F_n(X - \{p\}) = F_n(X) - F_n(p, X)$.

Given a map $f : X \to Y$ between continua and $n \in \mathbb{N}$, the function $f_n : F_n(X) \to F_n(Y)$ given by $f_n(A) = f(A)$ is the *induced map by* f *between the* n^{th}*- symmetric products of* X *and* Y. By [20, Corollary 1.8.23, p. 65], we have that f_n is a map and, it is easy to see that f_n is onto when f is onto. Therefore, if $n > 1$ and $K \in F_n(X)$, we can consider the natural induced function $f_{n,K} : F_n^K(X) \to F_n^{f(K)}(Y)$, which is defined by:

$$f_{n,K}([A]) = \begin{cases} \pi^Y_{n,f(K)}(f_n(A)) & \text{if } [A] \neq F^X_{n,K}, \\ F^Y_{n,f(K)} & \text{if } [A] = F^X_{n,K}. \end{cases}$$

By [7, Theorem 4.3, p. 126], we obtain that $f_{n,K}$ is a map.

3 Two Important Examples

In this section, we present the geometric model of $F_2^p(X)$ for each $p \in X$, when X is the interval I or the unit circumference S^1, which will be useful in further sections.

Example 3.1 The geometric model of $F_2^p(I)$.

Let $T = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$. It is well known that the map $\psi : F_2(I) \to T$ defined by $\psi(\{a, b\}) = (\min\{a, b\}, \max\{a, b\})$ is a homeomorphism (see [13, Section 13, p. 51]). We shall consider two cases.

Case I. $p \in \{0, 1\}$.
It is enough to consider the case $p = 0$. Since $\psi(F_2(0, I)) = \{(0, x) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$, it is easy to see that $T/\psi(F_2(0, I))$ is a 2-cell.

Case II. $p \in I - \{0, 1\}$.
Since $\psi(F_2(p, I)) = \{(x, p) \in \mathbb{R}^2 : 0 \leq x \leq p\} \cup \{(p, x) \in \mathbb{R}^2 : p \leq x \leq 1\}$, it is easy to see that $T/\psi(F_2(p, I))$ is homeomorphic to three 2-cells glued by a common point on its boundary.

Therefore, we have that $F_2^p(I)$ is homeomorphic to $F_2(I)$ when $p \in \{0, 1\}$ and $F_2^p(I)$ is homeomorphic to three 2-cells glued by a common point on its boundary in other case.

Example 3.2 The geometric model of $F_2^p(S^1)$.

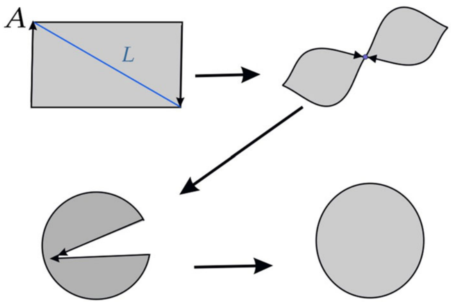
Let $p \in S^1$. It is well known that $F_2(S^1)$ is homeomorphic to the Möbius strip (see [2, p. 877] or [13, Section 14, p. 53]), which can be obtained from $A = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$, when we identify the point $(1, y)$ with the point $(-1, -y)$ for each $y \in [-1, 1]$, and since S^1 is homogeneous, we can suppose that p is represented in A by $(-1, 1)$. Therefore, under this assumption, we have that $F_2(p, S^1)$ is represented by $L = \{(x, -x) \in A : x \in [-1, 1]\}$. Hence, $F_2^p(S^1)$ is homeomorphic to A/L, which is a 2-cell (see Fig. 1).

4 The Connectivity Structure of the Hyperspace

In this section, we study the local connectedness and the arcwise connectedness of $F_n^K(X)$; also, we characterize the cut points of $F_n^K(X)$; moreover, if $[A] \in F_n^K(X)$ is a cut point, we characterize the components of $F_n^K(X) - \{[A]\}$.

Lemma 4.1 *Let X be a continuum, $n > 1$ and $K \in F_n(X)$. If $F_n(X) - F_n(K, X)$ is locally connected, then X is locally connected.*

Fig. 1 Geometrical model of $F_2^p(S^1)$



Proof Let $x \in X$ and let U be an open neighborhood of x in X. We shall consider two cases:

Case I. $K \neq \{x\}$.
Let $y \in X - K$ be such that $x \neq y$. Let $\epsilon > 0$ be such that $B_\epsilon(x) \subset U$, $B_\epsilon(x) \cap B_\epsilon(y) = \emptyset$, $B_\epsilon(y) \cap K = \emptyset$ and $B_\epsilon(x) \cap (K - \{x\}) = \emptyset$. Set $A = \{x, y\}$ and $\mathcal{U} = \langle B_\epsilon(x), B_\epsilon(y)\rangle_n$. Notice that $\mathcal{U}$ is an open subset of $F_n(X)$ and $A \in \mathcal{U} \subset F_n(X) - F_n(K, X)$.
Let $\mathcal{V}$ be a connected open subset of $F_n(X) - F_n(K, X)$, such that $A \in \mathcal{V} \subset \mathcal{U}$. By [5, Lemma 6.1, p. 851], we have that $\bigcup \mathcal{V}$ is an open subset of X, such that $(\bigcup \mathcal{V}) \cap \mathrm{B}_\epsilon(x)$ is connected. Thus, $(\bigcup \mathcal{V}) \cap \mathrm{B}_\epsilon(x)$ is an open connected neighborhood of x in U.

Case II. $K = \{x\}$.
By Case I, X is locally connected in each point of $X - \{x\}$, and thus, X is cik in each point of $X - \{x\}$. By [23, Corollary 5.13, p. 78], X must be cik at x; therefore, by [23, Exercise 5.22, p. 83], X is locally connected in x.

This finishes the proof of this lemma. □

Theorem 4.2 *Let X be a continuum, $n > 1$ and $K \in F_n(X)$. Then, X is locally connected if and only if $F_n^K(X)$ is locally connected.*

Proof Suppose that X is locally connected. By [17, Lemma 2, p. 286], $F_n(X)$ is locally connected. Then, $F_n^K(X)$ is locally connected, since it is the continuous image of a locally connected continuum.

Now, suppose that $F_n^K(X)$ is locally connected, by [16, Theorem 3, p. 230], we have that $F_n^K(X) - \{F_{n,K}^X\}$ is locally connected, and thus, by Remark 2.1, $F_n(X) - F_n(K, X)$ is locally connected, and then, the result follows from Lemma 4.1. This ends the proof of this theorem. □

Theorem 4.3 *Let X be a continuum, $n > 1$ and $K \in F_n(X)$. If X is arcwise connected, then $F_n^K(X)$ is an arcwise connected continuum.*

Proof Since X is an arcwise connected continuum, then $F_n(X)$ is an arcwise connected continuum (see [2, (a), p. 877]); therefore, the result follows from the fact that $F_n^K(X)$ is the continuous image of an arcwise connected space.

□

The following result shows that the converse of Theorem 4.3 is not true in general.

Theorem 4.4 *Let X be a compactification of $(0, 1]$ with remainder M and $n > 1$. If M is arcwise connected and $K \in F_{n-1}(M)$, then $F_n^K(X)$ is arcwise connected.*

Proof Let $[A] \in F_n^K(X) - \{F_{n,K}^X\}$. We will show that there exists an arcwise connected subset of $F_n^K(X)$ having $[A]$ and $F_{n,K}^X$.
Notice that $\pi_{n,K}^X(A) = [A]$. Suppose that $K = \{k_1, k_2, ..., k_m\}$, $(m < n)$. We shall consider three cases:

Case I. $A \subset M$.
Since M is arcwise connected, by [2, (a), p. 877], $F_n(M)$ is arcwise connected. Thus, $\pi_{n,K}^X(F_n(M))$ is an arcwise connected subset of $F_n^K(X)$ having $[A]$ and $F_{n,K}^X$.

Case II. $A \cap M \neq \emptyset$ and $A \cap (X - M) \neq \emptyset$.
Suppose that $A \cap M = \{a_1, a_2, ..., a_h\}$ and $A \cap (X - M) = \{b_1, b_2, ..., b_l\}$.
Let $x \in X - M$. Since $F_{n-1}(M)$ is arcwise connected, there exists an embedding $\alpha : [0, 1] \to F_{n-1}(M)$, such that $\alpha(0) = K$ and $\alpha(1) = \{a_1, a_2, ..., a_h\}$. Let $\alpha_1 : [0, 1] \to F_n(X)$ be given by $\alpha_1(t) = \{x\} \cup \alpha(t)$. It is clear that α_1 is a map, such that $\alpha_1(0) = K \cup \{x\}$ and $\alpha_1(1) = \{x, a_1, a_2, ..., a_h\}$.

Now, since $F_l(X - M)$ is arcwise connected, there exists an embedding $\beta : [0, 1] \to F_l(X)$, such that $\beta(0) = \{x\}$ and $\beta(1) = \{b_1, b_2, ..., b_l\}$. Let $\beta_1 : [0, 1] \to F_n(X)$ be given by $\beta_1(t) = \{a_1, a_2, ..., a_h\} \cup \beta(t)$. It is clear that β_1 is a map, such that $\beta_1(0) = \{x, a_1, a_2, ..., a_h\}$ and $\beta_1(1) = A$. Hence, $\pi_{n,K}^X(\alpha_1([0, 1]) \cup \beta_1([0, 1]))$ is an arcwise connected subset of $F_n^K(X)$ having $[A]$ and $F_{n,K}^X$.

Case III. $A \subset X - M$.
For each $i \in \{1, 2, ..., m\}$, let $\{k_j^i\}_{j=1}^{\infty}$ be a sequence of $X - M$, such that $\lim_{j\to\infty} k_j^i = k_i$ and $k_{j+1}^1 < k_j^m < k_j^{m-1} < \cdots < k_j^1$ with respect to the natural order induced by $(0, 1]$. For each $j \in \mathbb{N}$, let $K_j = \{k_j^m, k_j^{m-1}, ..., k_j^1\}$, $A_j^1 = K_j$, $A_j^{m+1} = K_{j+1}$, and for each $i = 2, \ldots, m$, let:

$$A_j^i = \{k_j^i, k_j^{i+1}, \ldots, k_j^m, k_{j+1}^1, k_{j+1}^2, \ldots, k_{j+1}^{i-1}\}.$$

Let $j \in \mathbb{N}$. For each $i = 1, 2, \ldots, m$, let $\alpha_j^i : [0, 1] \to X - M$ be an embedding, such that $\alpha_j^i(0) = k_j^i$ and $\alpha_j^i(1) = k_{j+1}^i$, and let $\beta_j^i, \gamma_j^i : [0, 1] \to F_n(X)$ be defined by $\beta_j^i(t) = A_j^i \cup \{\alpha_j^i(t)\}$ and $\gamma_j^i(t) = A_j^{i+1} \cup \{\alpha_j^i(t)\}$. It is easy to show that β_j^i and γ_j^i are maps, such that $\beta_j^i(0) = A_j^i$, $\beta_j^i(1) = A_j^i \cup \{k_{j+1}^i\} = \gamma_j^i(0)$ and $\gamma_j^i(1) = A_j^{i+1}$.
Notice that $\Gamma_j = \bigcup_{i=1}^m (\beta_j^i([0, 1]) \cup \gamma_j^i([0, 1]))$ satisfies the following:
(1) $\Gamma_j \subset F_n(X - M)$ for each $j \in \mathbb{N}$;
(2) for each $B \in \Gamma_j$, we have that $\{i : k_j^i \in B$ or $k_{j+1}^i \in B\} = \{1, \ldots, m\}$;
(3) $\bigcup_{l=1}^{\infty} \Gamma_l$ is an arcwise connected subset of $F_n(X - M)$ having K_j for each $j \in \mathbb{N}$.

Claim 1. $\mathrm{Cl}_{F_n(X)}(\bigcup_{l=1}^{\infty} \Gamma_l) - \bigcup_{l=1}^{\infty} \Gamma_l \subset F_n(K, X)$.
Let $E \in \mathrm{Cl}_{F_n(X)}(\bigcup_{l=1}^{\infty} \Gamma_l) - \bigcup_{l=1}^{\infty} \Gamma_l$. Without loss of generality, we can suppose that there exists a sequence $\{E_l\}_{l=1}^{\infty}$ contained in $\bigcup_{l=1}^{\infty} \Gamma_l$, such that $E_l \in \Gamma_l$ for each $l \in \mathbb{N}$ and $\lim_{l\to\infty} E_l = E$. Let $i \in \{1, 2, \ldots, m\}$. By (2), there exists a subsequence $\{k_{l_j}^i\}_{j=1}^{\infty}$ of $\{k_l^i\}_{l=1}^{\infty}$, such that $k_{l_j}^i \in E_l$. Hence, $k_i = \lim_{j\to\infty} k_{l_j}^i \in \lim_{l\to\infty} E_l = E$, and thus, $K \subset E$.

By the above, $\pi^X_{n,K}(\bigcup_{l=1}^{\infty} \Gamma_l \cup F_n(K, X))$ is an arcwise connected subset of $F_n^K(X)$. Now, by [2, (a), p. 877], we have that $F_n(X - M)$ is arcwise connected, and by (3), we know that $\bigcup_{j=1}^{\infty} \Gamma_j \subset F_n(X - M)$, since $A \in F_n(X - M)$, we conclude that A can be connected by an arc to a point of $\bigcup_{j=1}^{\infty} \Gamma_j$ in $F_n(X - M)$. Hence, there exists an arcwise connected subset of $F_n^K(X)$ having $[A]$ and $F^X_{n,K}$.
This finishes the proof of this theorem. □

Let Y be a compactification of $(0, 1]$ with remainder an arcwise connected continuum M, let X be a compactification of $(0, 1]$ with remainder Y, and let $K \in F_{n-1}(M)$. By Theorem 4.4, we have that $F_n^K(Y)$ is arcwise connected. From the fact that $F_n^K(Y) \subset F_n^K(X)$ and adapting the ideas that we use to prove Theorem 4.4, it can be shown that for each $[A] \in F_n^K(X) - \{F^X_{n,K}\}$, there exists an arcwise connected subset of $F_n^K(X)$ having $[A]$ and $F^X_{n,K}$. And thus, we can obtain that $F_n^K(X)$ is arcwise connected.

This means that the condition that the remainder M is arcwise connected in Theorem 4.4 is not necessary but useful. However, we will see that the remainder M cannot have a structure so complicated and the element K must be contained in the remainder.

Lemma 4.5 *Let X be a compactification of $(0, 1]$ with remainder M, $n > 1$ and $K \in F_{n-1}(X)$. Let $x \in M$ and $y \in X - M$ be such that $\{x\}, \{y\} \notin F_n(K, X)$. If Γ is an arc in $F_n^K(X)$ from $[\{x\}]$ to $[\{y\}]$, then $F^X_{n,K} \in \Gamma$.*

Proof Suppose that Γ is an arc from $[\{x\}]$ to $[\{y\}]$, such that $F^X_{n,K} \notin \Gamma$. By Remark 2.1, we have that $\alpha = (\pi^X_{n,K})^{-1}(\Gamma)$ is an arc having $\{x\}$ and $\{y\}$, and thus, by [5, Lemma 2.2, p. 819] and [6, Lemma 2.2, p. 252], $\bigcup \alpha$ is a locally connected subcontinuum in X having x and y. By the choice of x and y, we have that $\bigcup \alpha$ is homeomorphic to X, which is a contradiction. Hence, $F^X_{n,K} \in \Gamma$. □

Theorem 4.6 *Let X be a compactification of $(0, 1]$ with remainder M, $n > 1$ and $K \in F_{n-1}(X)$. If M is hereditarily indecomposable or $K \cap (X - M) \neq \emptyset$, then $F_n^K(X)$ is not arcwise connected.*

Proof Assume that M is hereditarily indecomposable and suppose that $F_n^K(X)$ is arcwise connected. Let $x \in M$ and $y \in X - M$ be such that $\{x\}, \{y\} \notin F_n(K, X)$. Let Γ be an arc in $F_n^K(X)$ from $[\{x\}]$ to $[\{y\}]$. By Lemma 4.5, $F^X_{n,K} \in \Gamma$. Let Γ' be an arc contained in $\Gamma - \{F^X_{n,K}\}$ having $[\{x\}]$. By Remark 2.1, we have that $\alpha = (\pi^X_{n,K})^{-1}(\Gamma')$ is an arc in $F_n(X)$ having $\{x\}$, and thus, by [5, Lemma 2.2, p. 819] and [6, Lemma 2.2, p. 252], $\bigcup \alpha$ is a locally connected subcontinuum in X having $\{x\}$, which contradicts the fact that M is hereditarily indecomposable. Therefore, $F_n^K(X)$ is not arcwise connected.

Now, assume that $K \cap (X - M) \neq \emptyset$ and suppose that $F_n^K(X)$ is arcwise connected. Let $k \in K \cap (X - M)$ and we denote by $\mathcal{C}_M$ the component of $X - \{k\}$ containing M. Let $x \in M$ and $y \in \mathcal{C}_M - M$ be such that $\{x\}, \{y\} \notin F_n(K, X)$. Let Γ be an arc in $F_n^K(X)$ from $[\{x\}]$ to $[\{y\}]$. By Lemma 4.5, $F^X_{n,K} \in \Gamma$. For convenience, we will provide the arc Γ with the natural order induced by $[0, 1]$ where $[\{x\}] < F^X_{n,K} < [\{y\}]$.

Let $[A_0] = \min\{[A] \in \Gamma : k \in A\}$. Notice that $[\{x\}] < [A_0] \leq F^X_{n,K}$. We define $\psi : \Gamma \to C_1(X)$ by $\psi([B]) = \bigcup(\pi^X_{n,K})^{-1}(\Gamma([\{x\}], [B]))$, where $\Gamma([\{x\}], [B])$ is the arc contained in Γ from $[\{x\}]$ to $[B]$. By [5, Lemma 2.2, p. 819] and [24, Lemma 1.48, p. 100], we have that ψ is a well-defined map. Since $k \notin \psi([B])$, for each $[\{x\}] \leq [B] < [A_0]$ and $\mathcal{C}_M \subset \psi([A_0])$, there exist $z \in \mathcal{C}_M \cap (X - M)$ and $[B_z] \in \Gamma$, such that $[\{x\}] < [B_z] < [A_0]$ and $z \in \psi([B_z])$. By [6, Lemma 2.2, p. 252], we obtain that $\psi([B_z])$ is a locally connected subcontinuum of X having x and z, which is a contradiction. Therefore, $F_n^K(X)$ is not arcwise connected. □

Theorem 4.7 *Let X be a continuum, $n > 1$ and $K \in F_n(X)$. Then, the following statements are equivalent:*

(1) X contains arcs.
(2) $F_n(X)$ contains arcs.
(3) $F_n^K(X)$ contains arcs.

Proof We shall prove (1) implies (3). Let α be an arc contained in X. Without loss of generality, we can suppose that $\alpha \cap K = \emptyset$. Since $F_1(\alpha) \subset F_n(X) - F_n(K, X)$, by Remark 2.1, $F_n^K(X)$ contains an arc.

We shall prove (3) implies (2). Let Γ be an arc in $F_n^K(X)$. Without loss of generality, we can suppose that $F_{n,K}^X \notin \Gamma$; by Remark 2.1, $F_n(X)$ contains an arc.
Finally, we shall prove (2) implies (1). Let Γ be an arc in $F_n(X)$. Then, by [6, Lemma 2.2, p. 252], $\bigcup \Gamma$ is a nondegenerate locally connected closed subset of X. Since it has at most n components, by [23, Theorem 8.23, p. 130], we have that $\bigcup \Gamma$ contains arcs. □

As a consequence of Theorem 4.7, we have that $F_n^K(X)$ is not arcwise connected in general, for each $n > 1$ and each $K \in F_n(X)$.

Theorem 4.8 *Let X be a continuum, $n > 1$ and $p \in X$. Then, p is a cut point of X if and only if $F_{n,p}^X$ is a cut point of $F_n^p(X)$.*

Proof Suppose that p is a cut point of X. Then, there are nonempty disjoint open subsets U and V of X, such that $X - \{p\} = U \cup V$. Let $\mathcal{U} = \langle U \rangle_n$ and let $\mathcal{V} = \langle X - \{p\}, V \rangle_n$. It is clear that $\mathcal{U}$ and $\mathcal{V}$ are nonempty disjoint open subsets of $F_n(X)$, such that $F_n(X) - F_n(p, X) = \mathcal{U} \cup \mathcal{V}$. By Remark 2.1, we have that $\pi_{n,p}^X(\mathcal{U})$ and $\pi_{n,p}^X(\mathcal{V})$ are nonempty disjoint open subsets of $F_n^p(X)$, such that $F_n^p(X) - \{F_{n,p}^X\} = \pi_{n,p}^X(\mathcal{U}) \cup \pi_{n,p}^X(\mathcal{V})$. Therefore, $F_{n,p}^X$ is a cut point of $F_n^p(X)$.

Now, suppose that $F_{n,p}^X$ is a cut point of $F_n^p(X)$. If $X - \{p\}$ were connected, then $F_n(X - \{p\})$ would be connected, and thus, by Remark 2.1, we obtain that $F_n^p(X) - \{F_{n,p}^X\}$ is connected, which is impossible. Therefore, $X - \{p\}$ is not connected. □

The following result shows that Theorem 4.8 is not true for $K \in F_n(X) - F_1(X)$.

Theorem 4.9 *Let X be a continuum and $n > 1$. If $K \in F_n(X) - F_1(X)$, then $F_n^K(X) - \{F_{n,K}^X\}$ is connected.*

Proof Let $K \in F_n(X) - F_1(X)$. By Remark 2.1, it is enough to show that $F_n(X) - F_n(K, X)$ is connected. Notice that $F_1(X) \subset F_n(X) - F_n(K, X)$. Let

$A = \{a_1, a_2, ..., a_m\} \in F_n(X) - F_n(K, X)$. We will show that there exists a connected subset $\mathcal{C}$ of $F_n(X) - F_n(K, X)$, such that $A \in \mathcal{C}$ and $\mathcal{C} \cap F_1(X) \neq \emptyset$. We shall consider three cases.

Case I. $A \cap K = \emptyset$.

For each $1 \leq i \leq m-1$, let $f_i : X \to F_n(X)$ be defined by $f_i(x) = \{x\} \cup \{a_{i+1}, a_{i+2}, ..., a_m\}$. By [24, Lemma 1.48, p. 100], we can obtain that f_i is a map, and thus, $f_i(X)$ is a connected subset of $F_n(X)$ containing $\{a_i, a_{i+1}, \ldots, a_m\}$ and $\{a_{i+1}, a_{i+2}, \ldots, a_m\}$. Therefore, $\mathcal{C} = \bigcup_{i=1}^{m-1} f_i(X)$ is a connected subset of $F_n(X)$, such that $A \in \mathcal{C}$ and $\mathcal{C} \cap F_1(X) \neq \emptyset$. By construction, we have that $\mathcal{C} \cap F_n(K, X) = \emptyset$.

Case II. $A \subset K$.

Let f_i be the map defined as in Case I for each $i \in \{1, 2, \ldots, m-1\}$. Since $f_i(X) \cap F_n(K, X) = \emptyset$, then $\mathcal{C} = \bigcup_{i=1}^{m-1} f_i(X)$ is a connected subset of $F_n(X)$, such that $A \in \mathcal{C}$, $\mathcal{C} \cap F_1(X) \neq \emptyset$ and $\mathcal{C} \cap F_n(K, X) = \emptyset$.

Case III. $A \not\subset K$ and $A \cap K \neq \emptyset$.

Let $A \cap K = \{a_1, a_2, \ldots, a_s\}$ and let $A - K = \{a_{s+1}, a_{s+2}, \ldots, a_m\}$. For each $1 \leq i \leq s$, let f_i be the map defined as in Case I. Since $f_i(X) \cap F_n(K, X) = \emptyset$, $\mathcal{C}_1 = \bigcup_{i=1}^{s} f_i(X)$ is a connected subset of $F_n(X)$, such that $A, A - K \in \mathcal{C}_1$ and $\mathcal{C}_1 \cap F_n(K, X) = \emptyset$. Since $(A - K) \cap K = \emptyset$, by Case I, there exists a connected subset $\mathcal{C}_2$ of $F_n(X)$, such that $A - K \in \mathcal{C}_2$, $\mathcal{C}_2 \cap F_1(X) \neq \emptyset$ and $\mathcal{C}_2 \cap F_n(K, X) = \emptyset$. Thus, $\mathcal{C}_1 \cup \mathcal{C}_2$ is a connected subset of $F_n(X)$ as we required.

This proves that for each $A \in F_n(X) - F_n(K, X)$, there exists a connected subset $\mathcal{C}$ of $F_n(X)$, such that $A \in \mathcal{C}$, $\mathcal{C} \cap F_n(K, X) = \emptyset$ and $\mathcal{C} \cap F_1(X) \neq \emptyset$. Therefore, we obtain that $F_n(X) - F_n(K, X)$ is connected. □

Theorem 4.10 *Let X be a continuum, $n > 1$ and $A, K \in F_n(X)$. Then, the following statements are equivalent:*

(1) $K = \{p\} \subset A$ and p is a cut point of X.
(2) $[A]$ is a cut point of $F_n^K(X)$.

Proof Suppose (2), we shall prove (1). Let $[A] \in F_n^K(X)$ be a cut point. Suppose that $[A] \neq F_{n,K}^X$, then $A \in F_n(X) - F_n(K, X)$, and by [17, Corollary 5, p. 289], we have that $F_n(X) - \{A\}$ is connected. Thus, $\pi_{n,K}^X(F_n(X) - \{A\}) = F_n^K(X) - \{[A]\}$ is a connected subset which is a contradiction. This shows that $[A] = F_{n,K}^X$, and therefore, we obtain that $K \subset A$. By Theorem 4.9, we have that $K = \{p\}$ for some $p \in X$, and by Theorem 4.8, we conclude that p is cut point.

The other implication follows from Theorem 4.8. □

To finish this section, we will count the components of $F_n^p(X) - \{F_{n,p}^X\}$, when p is a cut point of X.

Proposition 4.11 *Let X be a continuum and $p \in X$. If $X - \{p\}$ has m components, then $F_2^p(X) - \{F_{2,p}^X\}$ has $\frac{m(m+1)}{2}$ components.*

Proof Let $\{C_1, ..., C_m\}$ be the set of all components of $X - \{p\}$. By [21, Lemma 1, p. 204], $\langle C_i, C_j\rangle_2$ is connected for each $i, j \in \{1, \ldots, m\}$. Since $\langle C_i, C_j\rangle_2 \cap \langle C_h, C_l\rangle_2 = \emptyset$ if and only if $\{i, j\} \neq \{h, l\}$ and $F_2(X) - F_2(p, X) = \bigcup_{1 \leq i \leq j \leq m} \langle C_i, C_j\rangle_2$, then $\{\langle C_i, C_j\rangle_2 : 1 \leq i \leq j \leq m\}$ is the set of all components of $F_2(X) - F_2(p, X)$. Thus, by Remark 2.1, $\{\pi^X_{2,p}(\langle C_i, C_j\rangle_2) : 1 \le i \le j \le m\}$ is the set of all components of $F^p_2(X) - \{F^X_{2,p}\}$. Therefore, the number of components of $F^p_2(X) - \{F^X_{2,p}\}$ is equal to $\binom{m}{1} + \binom{m}{2} = \frac{m(m+1)}{2}$. □

Similar arguments can be used to prove the following result.

Proposition 4.12 *Let X be a continuum, $p \in X$, and $n > 1$. If $X - \{p\}$ has m components and $l = \min\{m, n\}$, then $F^p_n(X) - \{F^X_{n,p}\}$ has $\sum_{i=1}^{l} \binom{m}{i}$ components.*

Notice that, in Proposition 4.12, if $l = m$, then the numbers of components of $F^p_n(X) - \{F^X_{n,p}\}$ is $2^m - 1$.

5 Aposyndesis

In this section, we study the property of aposyndesis in the space $F^K_n(X)$.

Theorem 5.1 *Let X be a continuum, $n > 1$, and $K \in F_n(X)$. Then:*

(1) $F^K_n(X)$ is aposyndetic at $F^X_{n,K}$.
(2) If $[A] \in F^K_n(X) - \{F^X_{n,K}\}$, then $F^K_n(X)$ is aposyndetic at $[A]$ with respect to any point of $F^K_n(X) - \{F^X_{n,K}\}$.

Proof We shall prove (1). Let $[A] \in F^K_n(X) - \{F^X_{n,K}\}$. Let Γ and Λ be open subsets of $F^K_n(X)$, such that $F^X_{n,X} \in \Gamma$, $[A] \in \Lambda$ and $\Gamma \cap \Lambda = \emptyset$. Thus, $(\pi^X_{n,K})^{-1}(\Lambda)$ is an open subset of $F_n(X)$, such that $A \in (\pi^X_{n,K})^{-1}(\Lambda) \subset F_n(X) - F_n(K, X)$. By [17, Theorem 4, p. 289], there exists an open subset $\mathcal{U}$ of $F_n(X)$, such that $A \in \mathcal{U} \subset (\pi^X_{n,K})^{-1}(\Lambda)$ and $F_n(X) - \mathcal{U}$ is connected. Since $\mathcal{U} \cap F_n(K, X) = \emptyset$, by Remark 2.1, we have that $\pi^X_{n,K}(\mathcal{U})$ is an open subset of $F^K_n(X)$ and $[A] \in \pi^X_{n,K}(\mathcal{U}) \subset \Lambda$. Notice that $\pi^X_{n,K}(F_n(X) - \mathcal{U}) = F^K_n(X) - \pi^X_{n,K}(\mathcal{U})$ is connected. Thus, $F^K_n(X) - \pi^X_{n,K}(\mathcal{U})$ is a subcontinuum of $F^K_n(X)$, such that $F^X_{n,K} \in \Gamma \subset F^K_n(X) - \pi^X_{n,K}(\mathcal{U})$. Since $[A] \notin F^K_n(X) - \pi^X_{n,K}(\mathcal{U})$, we obtain that $F^K_n(X)$ is aposyndetic at $F^X_{n,K}$.
Similar arguments can be used to prove (2). □

Theorem 5.2 *Let X be a continuum and $n > 1$. If $K \in F_n(X) - F_1(X)$, then $F^K_n(X)$ is aposyndetic.*

Proof By Theorem 5.1, it is enough to show that for every element $[A] \in F^K_n(X) - \{F^X_{n,K}\}$, $F^K_n(X)$ is aposyndetic at $[A]$ with respect to $F^X_{n,K}$. Observe that $F_1(X) \subset F_n(X) - F_n(K, X)$.

Let $[A] \in F^K_n(X) - \{F^X_{n,K}\}$ with $A = \{a_1, \ldots, a_m\}$. For each $i \in \{1, \ldots, m\}$, let U_i be an open neighborhood of a_i in X, such that $(\mathrm{Cl}_X(U_i) - \{a_i\}) \cap K = \emptyset$, and we may assume that $U_i \cap U_j = \emptyset$ for any $i \ne j$. Notice that $\mathcal{U} = \langle U_1, \ldots, U_m\rangle_n$ is an open subset of $F_n(X)$ with the following properties:

$$A \in \mathcal{U} \subset F_n(X) - F_n(K, X) \text{ and } (\mathrm{Cl}_{F_n(X)}(\mathcal{U})) \cap F_n(K, X) = \emptyset. \quad (5.1)$$

Let $\mathcal{C}$ be a component of $\mathcal{U}$. By [5, Lemma 6.1, p. 851], $(\bigcup \mathcal{C}) \cap U_i$ is a connected subset of X for each $i \in \{1, \ldots, m\}$. For every $i = 1, \ldots, m$, let C_i be the component of U_i containing $(\bigcup \mathcal{C}) \cap U_i$. We shall show that $\mathcal{C} = \langle C_1, \ldots, C_m \rangle_n$. Let $B \in \mathcal{C}$. Notice that $(\bigcup B) \cap U_i \subset (\bigcup \mathcal{C}) \cap U_i \subset C_i$ and so, $\bigcup B \subset \bigcup_{i=1}^{m} C_i$. Since $B \in \mathcal{U}$, for each $i \in \{1, \ldots, m\}$, we have that $B \cap U_i \neq \emptyset$, and thus, $B \cap C_i \neq \emptyset$. This shows that $\mathcal{C} \subset \langle C_1, \ldots, C_m \rangle_n$. Since $\mathcal{C}$ is a component contained in $\langle C_1, \ldots, C_m \rangle_n$, and by [21, Lemma 1, p. 204], $\langle C_1, \ldots, C_m \rangle_n$ is a connected subset of $F_n(X)$ which is contained in $\mathcal{U}$. Hence, we conclude that $\mathcal{C} = \langle C_1, \ldots, C_m \rangle_n$.

On the other hand, by [23, Theorem 5.6, p. 74], we can choose $c_i \in \mathrm{Cl}_X(C_i) \cap \mathrm{Bd}_X(U_i)$ for each $i \in \{1, \ldots, m\}$. Thus, $\{c_1, \ldots, c_m\} \in F_n(X) - F(K, X)$ and $\{c_1, \ldots, c_m\} \cap K = \emptyset$.
For each $i \in \{1, \ldots, m-1\}$, let $f_i : X \to F_n(X)$ be defined by:

$$f_i(x) = \{x\} \cup \{c_{i+1}, \ldots, c_m\}.$$

By [24, Lemma 1.48, p. 100], we can obtain that f_i is a map, such that $\{c_i, \ldots, c_m\}$, $\{c_{i+1}, \ldots, c_m\} \in f_i(X)$ and $f_i(X) \cap F_n(K, X) = \emptyset$.

Thus, $\bigcup_{i=1}^{m-1} f_i(X)$ is a connected subset of $F_n(X)$, such that $(\bigcup_{i=1}^{m-1} f_i(X)) \cap F_n(K, X) = \emptyset$, $(\bigcup_{i=1}^{m-1} f_i(X)) \cap F_1(X) \neq \emptyset$ and $(\bigcup_{i=1}^{m-1} f_i(X)) \cap (\mathrm{Cl}_{F_n(X)}(\mathcal{C})) \neq \emptyset$. Let:

$$M_{\mathcal{C}} = \mathrm{Cl}_{F_n(X)}(\mathcal{C} \cup \left(\bigcup_{i=1}^{m-1} f_i(X) \right) \cup F_1(X)).$$

Note that $M_{\mathcal{C}}$ is a connected subset of $F_n(X)$, such that $M_{\mathcal{C}} \cap F_n(K, X) = \emptyset$. Let:

$$M = \mathrm{Cl}_{F_n(X)}(\bigcup \{M_{\mathcal{C}} : \mathcal{C} \text{ is a component of } \mathcal{U}\}).$$

Claim 1. $M \cap F_n(K, X) = \emptyset$.
Suppose that there exists $B \in M \cap F_n(K, X)$. Let $\{B_i\}_{i=1}^{\infty}$ be a sequence of $\bigcup\{M_{\mathcal{C}} : \mathcal{C}$ is a component of $\mathcal{U}\}$, such that $\lim B_i = B$. We shall show that there exists $N_1 \in \mathbb{N}$, such that $B_i \notin \bigcup_{j=1}^{m-1} f_j(X)$ for each $i \geq N_1$. On the contrary, suppose that there exists a subsequence $\{B_i\}_{i=1}^{\infty}$ in $\bigcup_{j=1}^{m-1} f_j(X)$. Then, $B \in \bigcup_{j=1}^{m-1} f_j(X)$, which is impossible, since $f_j(X) \cap F_n(K, X) = \emptyset$ for each $j \in \{1, \ldots, m-1\}$. Now, we shall show that there exists $N_2 \in \mathbb{N}$, such that $B_i \notin F_1(X)$ for each $i \geq N_2$. On the contrary, suppose that there exists a subsequence $\{B_i\}_{i=1}^{\infty}$ in $F_1(X)$. Then, $B \in F_1(X)$, which is impossible, since B has at least two points. By the above, we can assume that there exists $N \in \mathbb{N}$, such that $B_i \notin \bigcup_{i=1}^{m-1} f_i(X)$, $B_i \notin F_1(X)$, and thus, $B_i \in \mathrm{Cl}_{F_n(X)}(\mathcal{U})$ for each $i \geq N$. Then, $B \in \mathrm{Cl}_{F_n(X)}(\mathcal{U})$, which is a contradiction to the property of $\mathcal{U}$ in (5.1), and Claim 1 is proved.
Thus, M is a subcontinuum of $F_n(X)$ such that, by construction, $A \in \mathcal{U} \subset M$. Therefore, $A \in \mathrm{Int}_{F_n(X)}(M)$ and by Claim 1, $M \subset F_n(X) - F_n(K, X)$. Hence, $\pi_{n,K}^X(M)$ is a subcontinuum of $F_n^K(X)$, such that $[A] \in \mathrm{Int}_{F_n^K(X)}(\pi_{n,K}^X(M)) \subset \pi_{n,K}^X(M) \subset F_n^K(X) - \{F_{n,K}^X\}$. This shows that $F_n^K(X)$ is aposyndetic in $[A]$ with respect to $F_{n,K}^X$.
□

The following result shows that if $K \in F_1(X)$, then $F_n^K(X)$ is not necessary aposyndetic.

Theorem 5.3 *Let X be a compactification of $(0, 1]$ with non degenerate remainder Y, $n > 1$, and $x, p \in Y$ with $x \neq p$. Then $F_n^p(X)$ is not aposyndetic at $[\{x\}]$ with respect to $F_{n,p}^X$.*

Proof Let $\mathcal{M}$ be a subcontinuum of $F_n^p(X)$, such that $[\{x\}] \in \mathrm{Int}_{F_n^p(X)}(\mathcal{M})$. Since $\pi_{n,p}^X$ is a monotone map, we have that $M = (\pi_{n,p}^X)^{-1}(\mathcal{M})$ is a subcontinuum of $F_n(X)$, such that $\{x\} \in \mathrm{Int}_{F_n(X)}(M)$. By [5, Lemma 2.2, p. 819] and [5, Lemma 6.1, p. 851], we have that $\bigcup M$ is a subcontinuum of X, such that $x \in \mathrm{Int}_X(\bigcup M)$. Since $\mathrm{Int}_X(Y) = \emptyset$, we have that $Y \subset \bigcup M$. Hence, there exists $A \in M$, such that $p \in A$. Therefore, $F_{n,p}^X \in \mathcal{M}$. Thus, $F_n^p(X)$ is not aposyndetic. □

In the Euclidean plane, for each $m \in \mathbb{N}$, let $L_m = \{(t, \frac{t}{m}) : t \in [0, 1]\}$ and $L = [0, 1] \times \{0\}$. Set $W = (\bigcup_{m=1}^{\infty} L_m) \cup L$. Notice that W is an arcwise connected continuum. Similar arguments of the proof of Theorem 5.3 can be used to prove that if $p \in [0, 1) \times \{0\}$ and $n > 1$, then $F_n^p(W)$ is not aposyndetic.

Theorem 5.4 *Let X be a continuum, $n > 1$, and $K \in F_n(X)$. If X is aposyndetic, then $F_n^K(X)$ is aposyndetic.*

Proof By Theorem 5.2, it is enough to show that $F_n^p(X)$ is aposyndetic for each $p \in X$. Let $p \in X$ and $[A] \in F_n^p(X) - \{F_{n,p}^X\}$ with $A = \{a_1, ..., a_m\}$. By Theorem 5.1, it is enough to show that $F_n^p(X)$ is aposyndetic at $[A]$ with respect to $F_{n,p}^X$. Since X is aposyndetic, for each $i \in \{1, ..., m\}$, there exists a subcontinuum C_i of X, such that $a_i \in \mathrm{Int}_X(C_i) \subset C_i \subset X - \{p\}$. Notice that:

$$A \in \mathrm{Int}_{F_n(X)}(\langle C_1, ..., C_m \rangle_n) \subset \langle C_1, C_2, ..., C_m \rangle_n \subset F_n(X) - F_n(p, X).$$

We shall show that $\langle C_1, ..., C_m \rangle_n$ is a subcontinuum of $F_n(X)$. Since $X - \bigcup_{i=1}^m C_i$ and $X - C_i$ are open subsets of X for each $i = 1, \ldots, m$, it is easy to see that $\mathcal{H} = \{A \in F_n(X) : A \cap (X - \bigcup_{i=1}^m C_i) \neq \emptyset\}$ and $\mathcal{K} = \bigcup_{i=1}^m \{A \in F_n(X) : A \subset X - C_i\}$ are open subsets of $F_n(X)$. Then, $F_n(X) - \langle C_1, \ldots C_m \rangle_n = \mathcal{H} \cup \mathcal{K}$ is an open subset of $F_n(X)$. Therefore, $\langle C_1, \ldots C_m \rangle_n$ is a compact subset of $F_n(X)$. By [21, Lemma 1, p. 204], $\langle C_1, ..., C_m \rangle_n$ is a connected subset of $F_n(X)$. Hence, $\langle C_1, \ldots C_m \rangle_n$ is a subcontinuum of $F_n(X)$.

From the above, we have that $\pi_{n,p}^X(\langle C_1, ..., C_m \rangle_n)$ is a subcontinuum of $F_n^p(X)$, such that:

$$[A] \in \mathrm{Int}_{F_n^p(X)}(\pi_{n,p}^X(\langle C_1, ..., C_m \rangle_n))$$

and

$$\pi_{n,p}^X(\langle C_1, ..., C_m \rangle_n) \subset F_n^p(X) - \{F_{n,p}^X\}.$$

Therefore, $F_n^p(X)$ is aposyndetic. □

6 A Characterization of the Simple Closed Curve

In this section, we shall show that for each $p \in X$, $F_2^p(X)$ is a 2-cell if and only if X is a simple closed curve.

Recall that, given a continuum X, a nondegenerate subcontinuum A of X is called a *convergence continuum* of X, provided that there is a sequence $\{A_i\}_{i=1}^{\infty}$ of subcontinua A_i of X, such that $\lim_{i\to\infty} A_i = A$ and $A \cap A_i = \emptyset$ for each $i \in \mathbb{N}$ (see [23, Definition 5.11, p. 76]).

Lemma 6.1 *Let X be a continuum. If C is a convergence continuum of X and $p \in C$, then there exists a convergence continuum C_0 of X, such that $p \notin C_0$.*

Proof Since C is a convergence continuum of X, there exists a sequence of subcontinua $\{C_m\}_{m=1}^{\infty}$ of X, such that $\lim_{m\to\infty} C_m = C$ and $C_m \cap C = \emptyset$ for each $m \in \mathbb{N}$. Let $q \in C - \{p\}$ and let U be, an open subset of X, such that $q \in U$ and $p \notin \mathrm{Cl}_X(U)$. Notice that there exists a sequence $\{q_m\}_{m=1}^{\infty}$, such that $\lim_{m\to\infty} q_m = q$ and $q_m \in C_m$. Since $q \in U$, there exists $N \in \mathbb{N}$, such that for each $m \geq N$, $q_m \in U$. Now, for each $m \geq N$, let B_m be the component of U having q_m. Since $C(X)$ is compact (see [24, Theorem 0.8, p. 7]), we may assume without loss of generality that there exists $C_0 \in C(X)$, such that $\{\mathrm{Cl}_X(B_m)\}_{m=1}^{\infty}$ converges to C_0. By [23, Theorem 5.4, p. 73], for each $m \geq N$, there exists $r_m \in \mathrm{Cl}_X(B_m) \cap \mathrm{Bd}_X(U)$. Hence, $C_0 \cap \mathrm{Bd}_X(U) \neq \emptyset$. This shows that C_0 is nondegenerate and we obtain that C_0 is a convergence continuum. Finally, since $C_0 \subset \mathrm{Cl}_X(U)$, we conclude that $p \notin C_0$. □

Theorem 6.2 *Let X be a continuum and $p \in X$. If $F_2^p(X)$ is embedded in $\mathbb{R}^2$, then X is locally connected.*

Proof Suppose that X is not locally connected, and then, by [23, Theorem 5.12, p. 76] and [23, Exercise 5.22, p. 83], there exists a convergence continuum C of X. Let $\{C_m\}_{m=1}^{\infty}$ be a sequence of subcontinua of X, such that $\{C_m\}_{m=1}^{\infty}$ converges to C, $C_i \cap C = \emptyset$ for each $i \in \mathbb{N}$ and by [23, Exercise 5.23, p. 84], we can assume that $C_i \cap C_j = \emptyset$ for each $i, j \in \mathbb{N}$ with $i \neq j$. Notice that $F_2(C_m) \cap F_2(C) = \emptyset$ for each $m \in \mathbb{N}$. By Lemma 6.1, we can suppose that $p \notin C \cup (\bigcup_{m=1}^{\infty} C_m)$, and by Remark 2.1, we can assume that $F_2(C_m)$ and $F_2(C)$ are subsets of $\mathbb{R}^2$ for each $m \in \mathbb{N}$, and thus, $\dim(F_2(C)) \leq 2$. Since C is nondegenerate, then $\dim(C) \geq 1$. By [2, (b), p. 877], we have that $2 \leq \dim(C^2) \leq \dim(F_2(C))$. Hence, $\dim(F_2(C)) = 2$.
By [11, Theorem IV.3, p. 44], we have that $\mathrm{Int}_{\mathbb{R}^2}(F_2(C)) \neq \emptyset$. Since $\{C_m\}_{m=1}^{\infty}$ converges to C, we obtain $F_2(C_m) \cap F_2(C) \neq \emptyset$ for some $m \in \mathbb{N}$, which is a contradiction. Therefore, X is locally connected. □

Theorem 6.3 *Let X be a continuum. Then, there are two different points $p, q \in X$, such that $F_2^p(X)$ and $F_2^q(X)$ are embedded in $\mathbb{R}^2$ if and only if X is an arc or a simple closed curve.*

Proof Suppose that there are two different points $p, q \in X$, such that $F_2^p(X)$ and $F_2^q(X)$ are embedded in $\mathbb{R}^2$. By Theorem 6.2, X is locally connected, and by [23, Proposition 9.5, p. 142], it is enough to show that X does not contain simple triods. Suppose on the contrary that X contains a simple triod T. Without loss of generality, we can suppose that $p \notin T$, notice that the point q may be in T. Let J be an arc in X, such that $p \notin J$ and $J \cap T = \emptyset$, and thus, $\langle J, T\rangle_2 \subset F_2(X) - F_2(p, X)$ (if $q \in T$, then $\langle J, T\rangle_2 \cap F_2(q, X) \neq \emptyset$). Since $\langle J, T\rangle_2$ is homeomorphic to $J \times T$, by [3, Lemma 3.1, p. 58], $\langle J, T\rangle_2$ cannot be embedded in $\mathbb{R}^2$. Thus, by Remark 2.1, $\pi^X_{2,p}(\langle J, T\rangle_2) \subset F^p_2(X)$ cannot be embedded in $\mathbb{R}^2$, a contradiction. Therefore, X does not contain simple triods and we obtain that X is an arc or a simple closed curve. The other implication follows from Examples 3.1 and 3.2. □

Corollary 6.4 *Let X be a continuum. Then, $F^p_2(X)$ is a 2-cell for each $p \in X$ if and only if X is a simple closed curve.*

Proof This result follows from Theorem 6.3, Examples 3.1 and 3.2. □

7 Inverse Limits and Unicoherence

If $\{X_i\}_{i=1}^{\infty}$ is a sequence of topological spaces and $\{f_i\}_{i=1}^{\infty}$ is a sequence of onto maps, such that $f_i : X_{i+1} \to X_i$ for each $i \in \mathbb{N}$, the *inverse limit* of the inverse sequence $\{X_i, f_i\}_{i=1}^{\infty}$, denoted by $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, is the subspace of the product space $\prod_{i=1}^{\infty} X_i$ defined by:

$$\varprojlim\{X_i, f_i\}_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : f_i(x_{i+1}) = x_i \text{ for all } i\}.$$

It is known that when the spaces X_i are continua, then the inverse limit is a continuum (see [23, Theorem 2.4, p. 19]).

For each $\sigma = 1, 2, \ldots, \pi_\sigma : \varprojlim\{X_i, f_i\}_{i=1}^{\infty} \to X_\sigma$ denotes the σth projection map given by $\pi_\sigma((x_i)_{i=1}^{\infty}) = x_\sigma$.

Given $K \in F_n(X)$, we define the function $f_{n,K} : F^K_n(X) \to F^{f(K)}_n(Y)$ given by $f_{n,K}([A]) = \pi^Y_{n,f(K)}(f_n((\pi^X_{n,K})^{-1})([A]))$. By [7, Theorem 4.3, p. 126], we obtain that $f_{n,K}$ is a map and since f_n is onto when f is onto, it is not difficult to verify that $f_{n,K}$ is onto.

From [18, Corollary 6, p. 177] and [24, Theorem 1.169, p. 171], it is possible to show that if $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, then there exists a homeomorphism $h : \varprojlim\{F_2(X_i), (f_i)_2\}_{i=1}^{\infty} \to F_2(X)$, such that $A \in F_1(X)$ if and only if $A = h((\{x_i\})_{i=1}^{\infty})$ for some $(\{x_i\})_{i=1}^{\infty} \in \varprojlim\{F_2(X_i), (f_i)_2\}_{i=1}^{\infty}$. Thus, if $p \in X$, then $\{p\} = (\{p_i\})_{i=1}^{\infty} \in \varprojlim\{F_2(X_i), (f_i)_2\}_{i=1}^{\infty}$. Notice that $\pi_\sigma(F_2(p, X)) = F_2(p_\sigma, X_\sigma)$ for each $\sigma \in \mathbb{N}$. Hence, by [23, Lemma 2.6, p. 20], [18, Corollary 6, p. 177] and [9, Exercise 3.12.13, p. 233], we have the following lemma.

Lemma 7.1 *Let $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ and let $p = (\{p_i\})_{i=1}^{\infty} \in X$, where every X_i is a continuum. Then:*

(1) $F_2(p, X) = \varprojlim\{F_2(p_i, X_i), (f_i)_{2,p_{i+1}}|_{F_2(p_{i+1}, X_{i+1})}\}_{i=1}^{\infty}$ *and*

(2) $\varprojlim\{F^{p_i}_2(X_i), (f_i)_{2,p_{i+1}}\}_{i=1}^{\infty}$ *is homeomorphic to* $F^p_2(X)$.

Lemma 7.2 *Let X be a locally connected continuum and $p \in X$. If $f : F_2(X) \to S^1$ is a map, such that $f|_{F_2(p,X)}$ is inessential, then f is inessential.*

Proof Suppose that f is essential, and then, by [14, Lemma 1.9, p. 91], there exists a map $\sigma : S^1 \to F_2(X)$,, such that $f \circ \sigma$ is essential and $\sigma(1) = \{p\}$. By [15, Lemma 1.2, p. 13], there exist maps $\alpha, \beta : S^1 \to F_1(X)$, such that $\sigma(s) = \alpha(s) \cup \beta(s)$ for each $s \in S^1$. We define $\sigma_1, \sigma_2 : S^1 \to F_2(X)$ by:

$$\sigma_1(s) = \alpha(s) \cup \{p\} \text{ and } \sigma_2(s) = \beta(s) \cup \{p\}.$$

Let $F : S^1 \times S^1 \to F_2(X)$ be the map defined by $F(s,t) = \alpha(s) \cup \beta(t)$. Since $\sigma_1(S^1), \sigma_2(S^1) \subset F_2(p, X)$ and $f|_{F_2(p,X)}$ is inessential, we have that $f \circ \sigma_1$ and $f \circ \sigma_2$ are inessential. Then, there exists a map $h : S^1 \to \mathbb{R}$, such that $f \circ \sigma_1 = e \circ h$. Let $\pi : S^1 \times \{1\} \to S^1$ be the map given by $\pi(s,1) = s$, and we consider the map $h' : S^1 \times \{1\} \to \mathbb{R}$ defined by $h' = \pi \circ h$. Since $f \circ F|_{S^1 \times \{1\}} = e \circ h'$, we conclude that $f \circ F|_{S^1 \times \{1\}}$ is inessential. Similar arguments show that $f \circ F|_{\{1\} \times S^1}$ is inessential. By [14, Corollary 1.8, p. 91], $f \circ F$ is inessential. Then, there exists a map $r : S^1 \times S^1 \to \mathbb{R}$, such that $f \circ F = e \circ r$. Let $\Delta : S^1 \to S^1 \times S^1$ be the map given by $\Delta(s) = (s,s)$ and we consider the map $r' : S^1 \to \mathbb{R}$ defined by $r' = r \circ \Delta$. Since $f \circ \sigma = e \circ r'$, we conclude that $f \circ \sigma$ is inessential, which is a contradiction. This finishes the proof of this lemma. □

Theorem 7.3 *Let X be a locally connected continuum. If $p \in X$, then $F_2^p(X)$ is unicoherent.*

Proof Since X is locally connected, by Theorem 4.2, we have that $F_2^p(X)$ is locally connected. Hence, by [8, Theorem 3, p. 70], it is enough to show that $F_2^p(X)$ has the property (b), that is, every map $f : F_2^p(X) \to S^1$ is inessential.
Let $f : F_2^p(X) \to S^1$ and $\psi = f \circ \pi_{2,p}^X : F_2(X) \to S^1$ be maps. Since $\psi|_{F_2(p,X)}$ is constant, then $\psi|_{F_2(p,X)}$ is inessential, and we obtain that ψ is also inessential. Hence, there exists a map $h : F_2(X) \to \mathbb{R}$, such that $\psi = e \circ h$ and the following diagram is commutative:

$$\begin{array}{ccc} F_2(X) & \xrightarrow{\pi_{2,p}^X} & F_2^p(X) \\ h \downarrow & & f \downarrow \\ \mathbb{R} & \xrightarrow{e} & S^1 \end{array}$$

We shall prove that h is constant in the fibers of $\pi_{2,p}^X$. For this purpose, it is enough to show that h is constant in $F_2(p, X) = (\pi_{2,p}^X)^{-1}(F_{2,p}^X)$. Let $A, B \in F_2(p, X)$. Notice that $e(h(A)) = f(\pi_{2,p}^X(A)) = f(\pi_{2,p}^X(B)) = e(h(B))$, and hence, $h(F_2(p, X)) \subset e^{-1}(e(h(A)))$. Since $e^{-1}(e(h(A)))$ is a discrete space and $F_2(p, X)$ is connected, then $h(F_2(p, X))$ is a one point set. Therefore, h is constant in the fibers of $\pi_{2,p}^X$. By [7, Theorem 3.2, p. 123], there exists a map $\varphi : F_2^p(X) \to \mathbb{R}$, such that $h = \varphi \circ \pi_{2,p}^X$. Thus, $f \circ \pi_{2,p}^X = \psi = e \circ \varphi \circ \pi_{2,p}^X$. Notice that f and $e \circ \varphi$ have the same domain and range. Now, let $[A] \in F_2^p(X)$, and then, $[A] = \pi_{2,p}^X(B)$, for some $B \in F_2(X)$, and then, $f([A]) = f(\pi_{2,p}^X(B)) = e(\varphi(\pi_{2,p}^X(B))) = e(\varphi([A]))$, this shows that $f = e \circ \varphi$. Therefore, $F_2^p(X)$ is unicoherent. □

Theorem 7.4 *Let X be a continuum. If $p \in X$, then $F_2^p(X)$ is unicoherent.*

Proof By [10, p. 186] (see also, [23, Theorem 2.13, p. 24]), there exists an inverse sequence $\{X_i, f_i\}_{i=1}^{\infty}$, such that X_i is locally connected continuum for each $i \in \mathbb{N}$ and X is homeomorphic to $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. By Lemma 7.1, $F_2^p(X)$ is homeomorphic to:

$$\varprojlim\{F_2^{p_i}(X_i), (f_i)_{2,p_{i+1}}\}_{i=1}^{\infty}.$$

Since X_i is locally connected for each $i \in \mathbb{N}$, by Theorem 7.3, we have that $F_2^{p_i}(X_i)$ is unicoherent. And, from the fact, $(f_i)_{2,p_{i+1}}$ are onto maps for every $i \in \mathbb{N}$, by [25, Corollary 1, p. 228], we obtain that $F_2^p(X)$ is unicoherent . □

Theorem 7.5 *Let X be a continuum, $n > 2$ and $K \in F_n(X)$. Then $F_n^K(X)$ is unicoherent.*

Proof By [18, Theorem 8, p. 177], we have that $F_n(X)$ is unicoherent. Since $\pi_{n,K}^X$ is a monotone map, then $F_n^K(X)$ is unicoherent by [23, Corollary 13.35, p. 294]. □

8 Finite Graphs

In this section, we shall show that if X is a finite graph, then $F_2(X)$ is homeomorphic to $F_2^p(X)$ if and only if X is an arc or a simple n-od and p is an end point. Let X be a finite graph. For any $p, q \in X$ with $p \neq q$, pq denotes an arc from p to q in X. In this section, cone(X) denotes the topological cone over the continuum X (see [23, Definition 3.15, p. 41]).

Lemma 8.1 *Let X be a finite graph, let $p \in X$ be an end point and $q \in X - \{p\}$, such that the arc pq does not contain ramification points. If $\mathcal{U} = \langle pq, X\rangle_2$ and $\Lambda = \pi_{2,p}^X(\mathcal{U})$, then:*

(1) *$\mathcal{U}$ contains $F_2(p, X)$ in its interior,*
(2) *$\mathcal{U}$ is homeomorphic to $pq \times X$,*
(3) *Λ is homeomorphic to cone(X), where $F_{2,p}^X$ can be represented by the vertex of cone(X).*

Proof Let $W = pq - \{q\}$. Since $F_2(p, X) \subset \langle W, X\rangle_2 \subset \mathcal{U}$, then $\mathcal{U}$ contains $F_2(p, X)$ in its interior. Thus, (1) holds. We shall prove (2). Let $\mathcal{V} = \langle pq, X - W\rangle_2$. Notice that $\mathcal{U} = \mathcal{V} \cup \langle pq\rangle_2$. We define $\varphi_1 : \mathcal{V} \to pq \times X$ and $\varphi_2 : \langle pq\rangle_2 \to pq \times X$ by $\varphi_1(\{x, y\}) = (\{x, y\} \cap pq, \{x, y\} \cap (X - W))$, $\varphi_2(\{x, y\}) = (\max\{x, y\}, \min\{x, y\})$, respectively, and pq has the natural order induced by $[0, 1]$ where $q < p$. It is easy to show that φ_1 and φ_2 are embeddings. We define $\psi : \mathcal{U} \to pq \times X$ by:

$$\psi(\{x,y\}) = \begin{cases} \varphi_1(\{x,y\}) \text{ if } \{x,y\} \in \mathcal{V}, \\ \varphi_2(\{x,y\}) \text{ if } \{x,y\} \in \langle pq \rangle_2. \end{cases}$$

Notice that if $\{x, y\} \in \mathcal{V} \cap \langle pq \rangle_2$, then we can suppose that $y = q$, and thus, $\varphi_1(\{q, x\}) = \varphi_2(\{q, x\})$. Hence, ψ is a well-defined homeomorphism, which proves (2).

Since cone(X) is homeomorphic to $(pq \times X)/(\{p\} \times X)$ and $\psi(F_2(p, X)) = \{p\} \times X$, and then, cone$(X)$ is homeomorphic to $\psi(\mathcal{U})/\psi(F_2(p, X))$. Thus, we obtain that Λ is homeomorphic to cone(X), and by construction, $F^X_{2,p}$ can be represented by the vertex of cone(X), which proves (3). □

Theorem 8.2 *Let X be a simple n-od and $p \in X$. If p is an end point, then $F^p_2(X)$ is homeomorphic to $F_2(X)$.*

Proof Let $q \in X - \{p\}$ be such that the arc pq does not contain ramification points and let $W = pq - \{q\}$. By Lemma 8.1, $\mathcal{U} = \langle pq, X \rangle_2$ and $\Lambda = \pi^X_{2,p}(\mathcal{U})$ satisfies (1)-(3). Then, there exists a homeomorphism $h_1 : \mathcal{U} \to pq \times X$, such that $h_1(F_2(p, X)) = \{p\} \times X$. Since X is a simple n-od, by [26, Theorem 1, p. 79], there exists a homeomorphism $h_2 : pq \times X \to (pq \times X)/(\{p\} \times X)$, such that $h_2((q, x)) = [(q, x)]$ for each $x \in X - W$. By Lemma 8.1, there exists a homeomorphism $h_3 : (pq \times X)/(\{p\} \times X) \to \Lambda$, such that $h_3([\{p\} \times X]) = F^X_{2,p}$. We define $F : F_2(X) \to F^p_2(X)$ by:

$$F(A) = \begin{cases} \pi^X_{2,p}(A) \text{ if } A \in \mathrm{Cl}_{F_2(X)}(F_2(X) - \mathcal{U}), \\ h_3(h_2(h_1(A))) \text{ if } A \in \mathcal{U}. \end{cases}$$

Notice that, if $A \in \mathrm{Cl}_{F_2(X)}(F_2(X) - \mathcal{U}) \cap \mathcal{U}$, then $A = \{q, x\}$ for some $x \in X - W$ and $\pi_{2,p}(A) = h_3(h_2(h_1(A)))$. Hence, F is a well-defined homeomorphism. Therefore, $F_2(X)$ is homeomorphic to $F^p_2(X)$. □

Lemma 8.3 *Let X be a finite graph and let $p \in X$ be an end point. If $F_2(X)$ is homeomorphic to $F^p_2(X)$, then X has at most one ramification point.*

Proof Let $q_1, \ldots, q_n$ be the ramification points of X, such that ord$(q_i, X) = m_i \geq 3$ for each $i \in \{1, 2, \ldots, n\}$. Let $A_i = \{q_i\}$ and $B_{ij} = \{q_i, q_j\}$ with $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Let $C \in F_2(X) - \{A_i, B_{ij} : i, j \in \{1, 2, \ldots, n\}$ with $i \neq j\}$. By [3, Lemma 3.3, p. 60], there exists a neighborhood $\mathcal{U}$ in $F_2(X)$ having C, such that:

(a) $\mathcal{U}$ is a 2-cell, if $C \cap \{q_1, \ldots, q_n\} = \emptyset$ and
(b) $\mathcal{U}$ is homeomorphic to $T_{m_i} \times [0, 1]$ if $q_i \in C$, where $i \in \{1, ..., n\}$,

where T_{m_i} denotes a simple m_i-od. On the other hand, for every $k \in \{1, 2, \ldots, n\}$, q_k has a neighborhood T_{m_k} in X which is homeomorphic to a simple $m_k - od$, then A_k has a neighborhood in $F_2(X)$ which is $\langle T_{m_k} \rangle_2 = F_2(T_{m_k})$. Now, for every $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, by (f) of [3, Lemma 3.3, p. 60], there is a neighborhood $\mathcal{U}_{ij}$ in $F_2(X)$ of B_{ij}, such that $\mathcal{U}_{ij}$ is homeomorphic to cone$(K_{i,j})$, where $K_{i,j}$ denotes the complete bipartite graph. Notice that, for every $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$,

$A_i, B_{ij} \in F_2(X) - F_2(p, X)$, and thus, by Remark 2.1 and Lemma 8.1, $[A_i]$, $[B_{ij}]$, and $F^X_{2,p}$ have neighborhoods that are not 2-cell. Also, by [3, Lemma 3.1, p. 58], [3, Lemma 3.5, p. 61] and [3, Lemma 3.6, p. 62], the neighborhoods of $[A_i]$, $[B_{ij}]$ and $F^X_{2,p}$ cannot be embedded in $T_{m_i} \times [0, 1]$. Therefore, if $h : F_2^p(X) \to F_2(X)$ is a homeomorphism, by the above, we have that $h(\{[A_i], [B_{ij}], F^X_{2,p} : i, j \in \{1, 2, \ldots, n\}$ with $i \neq j\}) \subset \{A_i, B_{ij} : i, j \in \{1, 2, \ldots, n\}$ with $i \neq j\}$ which is impossible. Therefore, X contains at most one ramification point. □

Theorem 8.4 *Let X be a finite graph and let $p \in X$. Then, $F_2^p(X)$ is homeomorphic to $F_2(X)$ if and only if X is an arc or a simple n-od and p is an end point.*

Proof Suppose that $F_2(X)$ is homeomorphic to $F_2^p(X)$.
We shall show that p is an end point. On the contrary, suppose that p is not an end point. We will consider two cases.

Case I. p is a cut point.
By Theorem 4.8, $F^X_{2,p}$ is a cut point of $F_2^p(X)$, which is a contradiction to [17, Corollary 5, p. 289].

Case II. p is not a cut point.
Since X is a finite graph, by [27, (1.1), p. 64] and [27, Corollary 2, p. 79], X contains a simple closed curve, and thus, X is not unicoherent. Therefore, by [15, Lemma 1.1, p. 13], $F_2(X)$ is not unicoherent, contrary to Theorem 7.4.

This shows that p must be an end point.

Now, by Lemma 8.3, X has at most one ramification point, thus by [23, Proposition 9.5, p. 142] and [23, Lemma 9.9, p. 144], X is an arc or a simple closed curve or a simple $n-$od. Finally, by Example 3.2, X cannot be a simple closed curve. Therefore, X is either an arc or a simple $n-$od.

The other implication follows from Example 3.1 and Lemma 8.2. □

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