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Inverse limits of upper semi-continuous functions, connectedness and the Cantor set

José G. Anaya, Félix Capulín, Enrique Castañeda-Alvarado,
Mónica Sánchez-Garrido [*]

Universidad Autónoma del Estado de México, Facultad de Ciencias, Instituto Literiario 100, Col. Centro, Toluca, CP 50000, Mexico

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ABSTRACT

In this paper, we study inverse limits with one upper semi-continuous function which is the union of mappings. Using the concept of $Dom(F)$, we give sufficient conditions so that, the inverse limit is either a continuum or a Cantor set.

1. Introduction

In [7], W. S. Mahavier studied inverse limits using upper semi-continuous functions where he looked at inverse limits whose graphs are closed subsets of $[0,1] \times [0,1]$. In [4, Theorem 3.3], W. T. Ingram proved the following result.

Theorem 1.1. *If $\mathcal{F}$ is a finite collection of mappings from a continuum into itself, one of which is surjective and universal with respect to $\mathcal{F}$, and F is the set theoretic union of the elements of $\mathcal{F}$ which is an upper semi-continuous function, then $\varprojlim F$ is a continuum.*

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[*] Corresponding author.
E-mail addresses: jgao@uaemex.mx (J.G. Anaya), fcapulin@gmail.com (F. Capulín), eca@uaemex.mx (E. Castañeda-Alvarado), msanchezga830@profesor.uaemex.mx (M. Sánchez-Garrido).

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Also, W. T. Ingram proved the following result in [4, Theorem 4.2].

Theorem 1.2. *Let $\mathcal{F}$ be a collection of mappings from $[0,1]$ into itself. Suppose that $\mathcal{F}$ contains a mapping f with the following properties:*

1. *$f([0,1])$ is a non-degenerate set;*
2. *for each $g \in \mathcal{F}$, there exists a point $p_g \in f([0,1])$ such that $f(p_g) = g(p_g)$; and*
3. *for each $g \in \mathcal{F}$, $g(f([0,1])) = g([0,1])$.*

Let F be the function defined by the set theoretic union of the elements of $\mathcal{F}$. If F is an upper semi-continuous function, then $\varprojlim F$ is a one-dimensional continuum.

In [5], W. T. Ingram asked the following question [5, Problem 6.6]: what can be said about compacta that are inverse limits with a single upper semi-continuous function whose graph is the union of two graphs of mappings without coincidence points? In [3, Theorem 6.1], S. Greenwood, J. Kennedy and M. Lockyer showed that an inverse limit with a single upper semi-continuous function whose graph is the union of two graphs of mappings without coincidence points has $2^{\aleph_0}$ components. Here, we will give a partial answer to Ingram's question.

In [1, Definition 2], W. J. Charatonik and Ş. Şahan introduced the concept of $Dom(F)$, they proved that if X is a compact metric space and $F : X \to 2^X$ is an upper semi-continuous function, then $\varprojlim(X, F) = \varprojlim(Dom(F), F|_{Dom(F)})$. We use the concept of $Dom(F)$ and W. J. Charatonik and Ş. Şahan's results throughout the following to get as inverse limit either a continuum or a Cantor set.

In this paper, we study inverse limits with only one upper semi-continuous function F, defined by the set theoretic union of the elements of a family of mappings that are not necessarily surjective, and may or may not have coincidence points.

This paper is organized in the following way. In Section 2, we present basic definitions and auxiliary results. In Section 3, we provide some examples of inverse limits using one function F which is the set theoretic union of a family of mappings from $[0,1]$ into itself. In the proofs we apply the concept of $Dom(F)$. In Section 4, we give sufficient conditions for $\varprojlim(X, F)$ to be connected, and we give sufficient conditions for $\varprojlim(X, F)$ to be a Cantor set in the case where the mappings that define F are contractive.

2. Notation and auxiliary results

A *continuum* X is a non-degenerate connected compact metric space. A *mapping* is a continuous function. Let X and Y be topological spaces and let $f : X \to Y$ and $g : X \to Y$ be functions. A point $x \in X$ is called a *coincidence point* of f and g, if $f(x) = g(x)$. Let $\mathcal{F}$ be a collection of functions from X into Y. A function $f : X \to Y$ is *universal with respect to* $\mathcal{F}$ provided that f has a coincidence point for each element of $\mathcal{F}$. Let (X, d) and (Y, d') be metric spaces and let $f : (X, d) \to (Y, d')$ be a function. If there exists $\alpha_f \in [0, 1)$ such that $d'(f(x), f(y)) \leq \alpha_f d(x, y)$ for any $x, y \in X$, then f is called a *contractive function*. The number α_f is called a *module of* f. We say that a collection $\mathcal{F}$ of functions is *contractive* provided that each $f \in \mathcal{F}$ is a contractive function and $\alpha = \sup\{\alpha_f : \alpha_f \text{ is a module of } f,\ f \in \mathcal{F}\} < 1$. Also, α is called a *module of* $\mathcal{F}$.

We use the symbol $\mathbb{N}$ to denote the set of all positive integers. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of non-empty topological spaces. We use $\overline{x}$ to denote the point $(x_1, x_2, x_3, \dots)$ in $\prod\limits_{i=1}^{\infty} X_i$. Let $n \in \mathbb{N}$. We denote $(x_1, x_2, x_3, \dots, x_n) \in \prod\limits_{i=1}^{n} X_i$ by $\overline{x}_n$. Also, x_i is the i-th coordinate of the point $\overline{x}$ (or $\overline{x}_n$). Let $n \in \mathbb{N}$. The mapping $\pi_n : \prod_{i=1}^{\infty} X_i \to X_n$ defined by $\pi_n(\overline{x}) = x_n$ is called the *n-th projection*. If every X_i is a metric space with metric d_i bounded by 1, then we use the metric D for $\prod_{i=1}^{\infty} X_i$ defined by $D(\overline{x}, \overline{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$.

Let X be a metric compact space. Let 2^X be the collection of non-empty closed subsets of X endowed with the Hausdorff metric. Let X and Y be topological spaces. A function $F : X \to 2^Y$ is *upper semi-continuous at the point* $x \in X$, provided that if V is an open set in Y containing $F(x)$, then there exists an open set U in X containing x such that, if $t \in U$, then $F(t) \subseteq V$. F is *upper semi-continuous* if it is upper semi-continuous at every point of X. If $F : X \to 2^Y$ is a function, the set $G(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is the *graph* of F. Let $A \subseteq X$, we denote by $F(A)$ the set $\{y \in Y$: there is a point $x \in A$ such that $y \in F(x)\}$. The function F has a *surjective graph* or is *surjective*, provided $F(X) = Y$.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of non-empty compact metric spaces, and let $\{F_i\}_{i=1}^{\infty}$ be a sequence of functions such that for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ is upper semi-continuous. The sequence $\{X_i, F_i\}_{i=1}^{\infty}$ is called an *inverse sequence*. The *inverse limit* of the inverse sequence $\{X_i, F_i\}_{i=1}^{\infty}$ is the subspace of the topological product $\prod_{i=1}^{\infty} X_i$, given by

$$\{\overline{x} \in \prod_{i=1}^{\infty} X_i : x_i \in F_i(x_{i+1}), \text{ for each } i \in \mathbb{N}\}.$$

Let $\{X_i, F_i\}_{i=1}^{\infty}$ be an inverse sequence. We denote its inverse limit by $\varprojlim(X_i, F_i)$. In the case where the sequences $\{X_i\}_{i=1}^{\infty}$ and $\{F_i\}_{i=1}^{\infty}$ are constant sequences, i.e., there are a compact metric space X and a function $F : X \to 2^X$ such that $X_i = X$ and $F_i = F$ for each $i \in \mathbb{N}$, we denote the inverse limit by $\varprojlim F$.

Let $\{X_i, F_i\}_{i=1}^{\infty}$ be an inverse sequence, we adopt the following notation for each $n \in \mathbb{N}$:

1. $G'_n = \{\overline{x}_{n+1} \in \prod_{i=1}^{n+1} X_i : x_i \in F_i(x_{i+1}), 1 \leq i \leq n\}$.
   If the inverse sequence is a constant inverse sequence, then we consider the following sets:
2. $Dom_n(F) = \pi_1(G'_n)$.
3. $Dom(F) = \bigcap_{n=1}^{\infty} Dom_n(F)$.

Note that $Dom_{n+1}(F) \subseteq Dom_n(F)$ for each $n \in \mathbb{N}$.

Proposition 2.1. *Let X be a non-empty compact metric space. If $F : X \to 2^X$ is an upper semi-continuous function, then $Dom_n(F)$ and $Dom(F)$ are non-empty compact metric spaces.*

Proof. By [6, Theorem 110], we have that for each $n \in \mathbb{N}$, the set G'_n is a non-empty compact metric space. By [6, Theorem 111], we get that $\varprojlim F$ is a non-empty compact metric space. Since $Dom_n(F) = \pi_1(G'_n)$, $Dom(F) = \pi_1(\varprojlim F)$ and π_1 is a mapping, then $Dom_n(F)$ and $Dom(F)$ are non-empty compact metric spaces. □

From [2, Theorem 4.1], we have that if F is a mapping and X is a compact Hausdorff space, then $F(Dom(F)) = Dom(F)$. In the case where the function F is upper semi-continuous and X is compact metric space, we have from definition of $Dom(F)$, that $F(Dom(F)) = Dom(F)$.
Let $\mathcal{F}$ be a non-empty collection of mappings from X into itself and let $F : X \to 2^X$ be the function defined

by the set theoretic union of the elements of $\mathcal{F}$. If F is an upper semi-continuous function, it is not difficult to see that $Dom_1(F) = \bigcup_{f\in\mathcal{F}} f(X)$, and $Dom_{n+1}(F) = \bigcup_{f\in\mathcal{F}} f(Dom_n(F))$.

The following result is the main tool used in this paper.

Theorem 2.2. *[1, Theorem 3]. Let X be a non-empty compact metric space and let $F : X \to 2^X$ be an upper semi-continuous function. Then $\varprojlim(X, F) = \varprojlim(Dom(F), F|_{Dom(F)})$.*

The following theorem is a consequence of Theorem 2.2 and the definition of the inverse limit.

Theorem 2.3. *Let X be a non-empty compact metric space and let $F : X \to 2^X$ be an upper semi-continuous function. If $Dom(F)$ is a totally disconnected space, then $\varprojlim F$ is a totally disconnected space.*

A topological space X is *perfect* if every point of X is an accumulation point of X.

Theorem 2.4. *Let X be a non-empty compact metric space and let $\mathcal{F}$ be a non-empty collection of mappings from X into itself. Let $F : X \to 2^X$ be the function defined by the set theoretic union of the elements of $\mathcal{F}$. If F is upper semi-continuous function and $Dom(F)$ is a perfect space, then $\varprojlim F$ is a perfect space.*

Proof. Let $\overline{x} \in \varprojlim(Dom(F), F|_{Dom(F)})$. Let

$$U = \left(U_1 \times U_2 \times U_3 \times \cdots \times U_n \times \prod_{i=n+1}^{\infty} Dom(F)\right) \cap \varprojlim F$$

be a basic open neighbourhood of $\overline{x}$ in $\varprojlim F$, where U_i is an open neighbourhood of x_i in $Dom(F)$ for each $1 \le i \le n$. Now, for every $1 \le i \le n-1$, we consider $f_i \in \mathcal{F}$ such that $x_i = f_i(x_{i+1})$. Let $W_1 = U_1$. By continuity of f_1, there exists an open neighbourhood W_2 such that $W_2 \subseteq U_2$, $f_1(W_2) \subseteq W_1$ and $x_2 \in W_2$. If $1 \le i \le n-1$ and W_i has been defined, then there exists an open neighbourhood W_{i+1} such that $W_{i+1} \subseteq U_{i+1}$, $f_i(W_{i+1}) \subseteq W_i$ and $x_{i+1} \in W_{i+1}$.

Since $Dom(F)$ is a perfect space and W_n is an open subset of $Dom(F)$, then there exists $y_n \in W_n \cap Dom(F)$ such that $y_n \neq x_n$. Note that there exists $y_{n-1} \in W_{n-1}$ such that $y_{n-1} = f_{n-1}(y_n)$. Moreover, for each $1 \le i < n-1$ there exists $y_i \in W_i$ such that $y_i = f_i(y_{i+1})$. Hence, $\overline{x}_n, \overline{y}_n \in W_1 \times W_2 \times W_3 \times \cdots \times W_n$.

Since $y_n \in Dom(F)$, there exists $\overline{z} \in \varprojlim F$ such that $\pi_1(\overline{z}) = y_n$. Let $\overline{w} = (y_1, y_2, \ldots, y_n, z_2, z_3, \ldots) \in \varprojlim F$. Thus, $\overline{w} \in U$ and $\overline{x} \neq \overline{w}$. Therefore, $\varprojlim(Dom(F), F|_{Dom(F)})$ is a perfect space. □

The result below follows from [6, Theorem 111], Theorem 2.3 and Theorem 2.4.

Corollary 2.5. *Let X be a non-empty compact metric space, let $\mathcal{F}$ be a non-empty collection of mappings from X into itself and let $F : X \to 2^X$ be the function defined by the set theoretic union of the elements of $\mathcal{F}$. If F is upper semi-continuous function and $Dom(F)$ is a Cantor set, then $\varprojlim F$ is homeomorphic to the Cantor set.*

3. Examples

In this section, we present examples of inverse limits with a single upper semi-continuous function F which is the union of a family of mappings from $[0, 1]$ into itself. In the proofs we use the concept of $Dom(F)$.

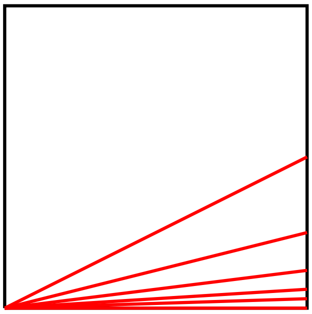

Fig. 1. The graph of the function F in Example 3.1.

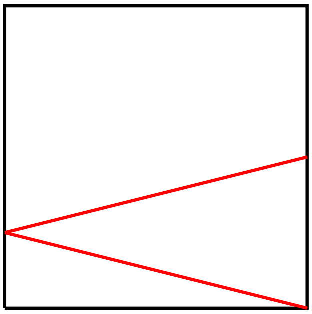

Fig. 2. The graph of the function F in Example 3.3.

Example 3.1. Let $f_n : [0,1] \to [0,1]$ be a mapping given by $f_n(x) = \frac{x}{2^n}$ for each $n \in \mathbb{N}$. If $F : [0,1] \to 2^{[0,1]}$ is the upper semi-continuous function defined by $F(x) = \{0\} \cup \{f_n(x) : n \in \mathbb{N}\}$, then $\varprojlim F = \{(0,0,0,\ldots)\}$ (see Fig. 1 for the graph of F).

Proof. Note that for each $n \in \mathbb{N}$, $Dom_n(F) = [0, \frac{1}{2^n}]$. Thus, $Dom(F) = \{0\}$, and so that $\varprojlim(Dom(F), F|_{Dom(F)}) = \{(0,0,\ldots)\}$. By Theorem 2.2, $\varprojlim F = \{(0,0,\ldots)\}$. $\square$

Example 3.2. Let $f, g : [0,1] \to [0,1]$ be mappings defined by $f(x) = \frac{x}{2}$ and $g(x) = \frac{1-x}{2}$. Let $F : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function given by $F(x) = \{f(x), g(x)\}$. Then $\varprojlim F$ is a continuum.

Proof. Note that for each $n \in \mathbb{N}$, $Dom_n(F) = [0, \frac{1}{2}]$. Thus, $Dom(F) = [0, \frac{1}{2}]$. Since $F|_{[0,\frac{1}{2}]} : [0, \frac{1}{2}] \to 2^{[0,\frac{1}{2}]}$ is surjective and it has connected graph, by [8, Theorem 3.1], we have that $\varprojlim \left(\left[0, \frac{1}{2}\right], F|_{[0,\frac{1}{2}]} \right)$ is a continuum. By Theorem 2.2, $\varprojlim F$ is a continuum. $\square$

In the following example we will prove that $Dom(F)$ is homeomorphic to the Cantor set and hence the inverse limit is homeomorphic to the Cantor set.

Example 3.3. Let $F : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function given by $F(x) = \{\frac{x+1}{4}, \frac{1-x}{4}\}$. Then $\varprojlim F$ is homeomorphic to the Cantor set (see Fig. 2 for the graph of F).

Proof. Note that $Dom_1(F) = [0, \frac{1}{2}]$, $Dom_2(F) = [\frac{1}{8}, \frac{3}{8}]$ and $Dom_3(F) = [\frac{5}{32}, \frac{7}{32}] \cup [\frac{9}{32}, \frac{11}{32}]$. From here, for each $n \geq 2$, the set $Dom_{n+1}(F)$ is the union of twice as many disjoint intervals as $Dom_n(F)$. Observe that the intervals form nested sequences whose lengths limit to zero. Hence, $Dom(F)$ consists of $2^{\aleph_0}$ singleton

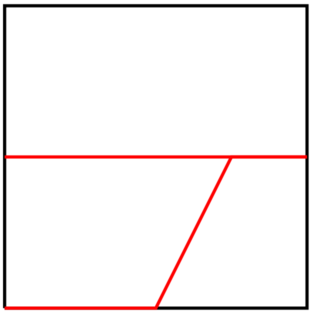

Fig. 3. The graph of the function H_α in Example 3.5.

components. Moreover, $Dom(F)$ is obtained by removing "middles thirds" and so it is a Cantor set. By Corollary 2.5, $\varprojlim F$ is homeomorphic to the Cantor set. $\square$

The following examples show that the converse of Corollary 2.5 is not true. We consider four cases. We give functions whose graphs are:

1. not connected nor surjective, with $|Dom(F)| = 2$;
2. connected and not surjective, with $|Dom(F)| = 2$;
3. not connected and surjective, with $Dom(F) = [0, 1]$;
4. not connected nor surjective, such that $Dom(F)$ is a not connected one-dimensional set.

Example 3.4. Let $c, d \in [0, 1]$ be such that $0 \leq c < d \leq 1$. Let $F : [0, 1] \to 2^{0,1]}$ be the upper semi-continuous function defined by $F(x) = \{c, d\}$. Then $\varprojlim F$ is homeomorphic to the Cantor set.

Proof. Clearly, $\varprojlim F = \{\overline{x}\colon \text{for each } i \in \mathbb{N}, x_i = c \text{ or } x_i = d\}$, then $\varprojlim F$ is homeomorphic to the Cantor set. $\square$

Note that [5, Example 1.2] is the particular case of Example 3.4 when $c = 0$ and $d = 1$.

In the following example we define a family $\{H_\alpha : \alpha \in [1, \infty)\}$ of upper semi-continuous non-surjective functions each one having connected graph and such that for each fixed real number $\alpha \in [1, \infty)$, we have that $\varprojlim H_\alpha$ is homeomorphic to the Cantor set. Moreover, each H_α is defined as the union of two mappings from $[0, 1]$ into itself having coincidence points.

Example 3.5. Let $\alpha \in [1, \infty)$ and let $H_\alpha : [0, 1] \to 2^{[0,1]}$ be the upper semi-continuous function defined by $H_\alpha(x) = \{g(x), f_\alpha(x)\}$, where $g(x) = \frac{1}{2}$ and

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ \alpha\left(x - \frac{1}{2}\right) & \text{if } x \in [\frac{1}{2}, \frac{\alpha+1}{2\alpha}], \\ \frac{1}{2} & \text{if } x \in [\frac{\alpha+1}{2\alpha}, 1]. \end{cases}$$

Then $\varprojlim H_\alpha$ is homeomorphic to the Cantor set (see Fig. 3 for the graph of H_α).

Proof. Because $Dom_1(H_\alpha) = [0, \frac{1}{2}]$, and $Dom_n(H_\alpha) = \{0, \frac{1}{2}\}$ for $n > 1$, the set $Dom(H_\alpha) = \{0, \frac{1}{2}\}$. Observe that $\varprojlim(Dom(H_\alpha), H_\alpha|_{Dom(H_\alpha)}) = \prod_{i=1}^{\infty}\left\{0, \frac{1}{2}\right\}$. So, $\varprojlim H_\alpha$ is homeomorphic to the Cantor set. $\square$

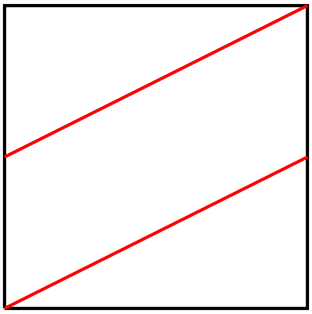

Fig. 4. The graph of the function F in Example 3.6.

In the following example, we define an upper semi-continuous surjective function, with a disconnected graph, as the union of two mappings without coincidence points. Again, $\varprojlim F$ is the Cantor set.

Example 3.6. Let $F : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function defined by $F(x) = \{f(x), g(x)\}$ where $f(x) = \frac{x}{2}$ and $g(x) = \frac{x+1}{2}$. Then $\varprojlim F$ is homeomorphic to the Cantor set (see Fig. 4 for the graph of F).

Proof. By [5, Theorem 1.6], we just need to prove that the inverse limit is a perfect and totally disconnected space.

Note that $G'_1 = \{(f(x), x) : x \in [0,1]\} \cup \{(g(x), x) : x \in [0,1]\}$ and the diameter of each component is $\frac{1}{2}$. Moreover, G'_n has 2^n components, each one of them of diameter $\frac{n+1}{2^{n+1}}$. The fact that $\lim\limits_{n\to\infty} \dfrac{n+1}{2^{n+1}} = 0$, implies that each component of $\displaystyle\bigcap_{n=1}^{\infty} \left(G'_n \times \prod_{i=n+2}^{\infty} [0,1] \right)$ has a single point. Since $\displaystyle\varprojlim F = \bigcap_{n=1}^{\infty} \left(G'_n \times \prod_{i=n+2}^{\infty} [0,1] \right)$, then $\varprojlim F$ is a totally disconnected space.

Since $Dom(F) = [0,1]$, then Theorem 2.4 implies that $\varprojlim F$ is a perfect space. □

Now, we will provide an example such that the graph of F is neither connected nor surjective, $Dom(F)$ is a one-dimensional disconnected set and $\varprojlim F$ is homeomorphic to the Cantor Set.

Example 3.7. Let $F : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function defined by $F(x) = \{f(x), g(x)\}$ where:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{1}{4}], \\ \frac{1}{8} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ \frac{2x-1}{4} & \text{if } x \in [\frac{3}{4}, 1], \end{cases}$$

and

$$g(x) = \begin{cases} \frac{2x+3}{4} & \text{if } x \in [0, \frac{1}{4}], \\ \frac{7}{8} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ \frac{x+1}{2} & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Then $\varprojlim F$ is homeomorphic to the Cantor set (see Fig. 5 for the graph of F).

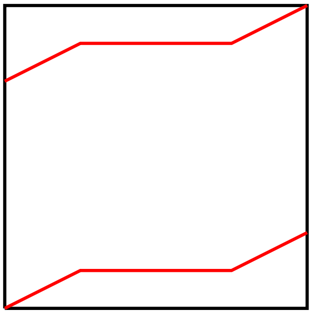

Fig. 5. The graph of the function F in Example 3.7.

Proof. Note that $G'_1 = \{(f(x), x) : x \in [0,1]\} \cup \{(g(x), x) : x \in [0,1]\}$ and the diameter of each component is $\frac{3}{8}$. Moreover, G'_n is the union of 2^n components, each one of them of diameter $\frac{n+2}{2^{n+2}}$. Hence, each component of $\bigcap_{n=1}^{\infty} \left(G'_n \times \prod_{i=n+2}^{\infty} [0,1] \right)$ is a single point. Therefore, $\varprojlim F$ is a totally disconnected space.

Observe that for each $n \in \mathbb{N}$, $Dom_n(F) = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, $Dom(F) = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$. Hence, Theorem 2.4 implies that $\varprojlim F$ is a perfect space. □

4. Main results

In this section, we will give sufficient conditions so that the inverse limit is either a continuum or a Cantor set.

In this section, we assume that X is a compact metric space with bounded metric, $\mathcal{F}$ is a non-empty countable collection of mappings from X into itself, and the function $F : X \to 2^X$ given by $F(x) = \{f(x) : f \in \mathcal{F}\}$ is upper semi-continuous not necessarily surjective and such that $Dom(F)$ is a non-degenerate set.

Now, we consider the following notation. For all $h, g \in \mathcal{F}$, $h \neq g$, let $P_{hg} = \{x \in X : h(x) = g(x)\}$, $P = \bigcup_{h,g \in \mathcal{F}} P_{hg}$ and $Q = \bigcap_{h,g \in \mathcal{F}} P_{hg}$.

Theorem 4.1. *Let X be a continuum. If for any $h, g \in \mathcal{F}$, $h \neq g$, $P_{hg} \cap Dom(F) \neq \emptyset$, then $\varprojlim F$ is a continuum. In particular, if $Q \cap Dom(F) \neq \emptyset$, then $\varprojlim F$ is a continuum.*

Proof. We will prove by induction that $Dom_n(F)$ is a connected space for each $n \in \mathbb{N}$. Since $P_{hg} \cap Dom(F) \neq \emptyset$ for each $h, g \in \mathcal{F}$, then $h(X) \cap g(X) \neq \emptyset$. Because h and g are mappings and X is a continuum, we have that $h(X)$ and $g(X)$ are connected. Hence, $h(X) \cup g(X)$ is a connected space. So, $\bigcup_{h,g \in \mathcal{F}} (h(X) \cup g(X))$ is a connected space. Since $Dom_1(F) = \bigcup_{f \in \mathcal{F}} f(X) = \bigcup_{h,g \in \mathcal{F}} (h(X) \cup g(X))$, then $Dom_1(F)$ is a connected space.

We suppose that $Dom_n(F)$ is a connected space. Since for each $h, g \in \mathcal{F}$, $P_{hg} \cap Dom_n(F) \neq \emptyset$, then $h(Dom_n(F)) \cap g(Dom_n(F)) \neq \emptyset$. Because h and g are mappings and $Dom_n(F)$ is a continuum, we have that $h(Dom_n(F))$ and $g(Dom_n(F))$ are connected. Thus, $h(Dom_n(F)) \cup g(Dom_n(F))$ is a connected space for each $h, g \in \mathcal{F}$. Then $\bigcup_{h,g \in \mathcal{F}} (h(Dom_n(F)) \cup g(Dom_n(F)))$ is a connected space. Since $Dom_{n+1}(F) = \bigcup_{f \in \mathcal{F}} f(Dom_n(F)) = \bigcup_{h,g \in \mathcal{F}} (h(Dom_n(F)) \cup g(Dom_n(F)))$, then $Dom_{n+1}(F)$ is a connected space.

Notice that $Dom_{n+1}(F) \subseteq Dom_n(F)$ for each $n \in \mathbb{N}$, and $Dom(F) = \bigcap_{n \in \mathbb{N}} Dom_n(F)$. So, $Dom(F)$ is a connected space. From Proposition 2.1, $Dom(F)$ is a compact space. This implies that $Dom(F)$ is a continuum. Because of $F|_{Dom(F)} : Dom(F) \to 2^{Dom(F)}$ has a surjective and connected graph, from [8, Theorem 3.1] we have that $\varprojlim(Dom(F), F|_{Dom(F)})$ is a continuum. Theorem 2.2 implies that $\varprojlim F$ is a continuum. $\square$

The following two results can be proved using the same ideas given in the proof of Theorem 4.1.

Theorem 4.2. *Let X be a continuum, let $h : X \to X$ be a mapping, and let $F : X \to 2^X$ be an upper semi-continuous function given by $F(x) = \{f(x) : f \in \mathcal{F}\} \cup \{h(x)\}$. Let P_f be the set of coincidence points of f and h, for each $f \in \mathcal{F}$. If $P_f \cap Dom(F) \neq \emptyset$, for each $f \in \mathcal{F}$, then $\varprojlim F$ is a continuum.*

Proposition 4.3. *Let X be a continuum, let $k \in \mathbb{N}$, $\mathcal{F} = \{f_i : f_i \text{ is a mapping from } X \text{ into itself}, 1 \leq i \leq k\}$ and $Q_i = \{x \in X : f_i(x) = f_{i+1}(x)\}$ for every $1 \leq i < k$. If $Q_i \cap Dom(F) \neq \emptyset$, for each $1 \leq i < k$, then $\varprojlim F$ is a continuum.*

Corollary 4.4. *Let X be a continuum. If there exists a point $p \in X$ such that $f(p) = p$ for each $f \in \mathcal{F}$, then $\varprojlim F$ is a continuum.*

Proof. Since for each $f \in \mathcal{F}$, $f(p) = p$, then $f^n(p) = p$ for each $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, $p \in \bigcap\limits_{f \in \mathcal{F}} f^n(X) \subseteq Dom_n(F)$. Hence, $p \in Dom(F)$. Fix $h \in \mathcal{F}$, then $p \in P_f$ for each $f \in \mathcal{F}$. By Theorem 4.2, $\varprojlim F$ is a continuum. $\square$

Note that condition 3 in Theorem 1.2, is replaced in Theorem 4.2 by a more general condition. Observe that Example 3.2 does not satisfy condition 3 in Theorem 1.2. However, it satisfies the hypothesis of Theorem 4.2. So, the inverse limit is a continuum. On the other hand, the function h in Theorem 4.2 is universal with respect to the family $\mathcal{F}$, but the mapping h is not assumed surjective. If h is surjective, then Theorem 1.1 is a Corollary of Theorem 4.2.

Examples 3.3 and 3.5 show that Theorem 4.1 and Theorem 4.2 are not always true if $P_{fg} \cap Dom(F) = \emptyset$ for some $f, g \in \mathcal{F}$ such that $f \neq g$ or $P_f \cap Dom(F) = \emptyset$ for some $f \in \mathcal{F}$.

4.1. Contractive mappings

We will give some notations for this section. Let $\mathcal{S}$ be the collection of infinite sequences $\{f_i\}_{i=1}^{\infty}$ of $\mathcal{F}$. For every $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the collection of all finite sequences $\{f_i\}_{i=1}^{n}$ of $\mathcal{F}$. For each $\mathbf{f} \in \mathcal{S}_n$, let $G'_n(\mathbf{f}) = \{\overline{x}_{n+1} \in \prod\limits_{i=1}^{n+1} X : x_i = f_i(x_{i+1}), 1 \leq i \leq n\}$. Note that for every $n \in \mathbb{N}$, $G'_n = \bigcup\limits_{\mathbf{f} \in \mathcal{S}_n} G'_n(\mathbf{f})$. For each $\overline{\mathbf{f}} \in \mathcal{S}$, let $\varprojlim \overline{\mathbf{f}} = \{\overline{x} \in \prod\limits_{i=1}^{\infty} X : x_i = f_i(x_{i+1}), \text{ for all } i \in \mathbb{N}\}$. On the other hand, given $\overline{\mathbf{f}} \in \mathcal{S}$ and $n \in \mathbb{N}$ let $\mathbf{f}_n = \{f_i : 1 \leq i \leq n\}$. Finally, if $\overline{\mathbf{f}} \in \mathcal{S}$, let $S(\overline{\mathbf{f}}) = \{\mathbf{f}_n : n \in \mathbb{N}\}$.

Proposition 4.5. $\varprojlim F = \bigcup\limits_{\overline{\mathbf{f}} \in \mathcal{S}} \varprojlim \overline{\mathbf{f}}$, *and for any* $\overline{\mathbf{f}} \in \mathcal{S}$,

$$\varprojlim \overline{\mathbf{f}} = \bigcap_{n \in \mathbb{N}} \{G'_n(\mathbf{f}_n) \times \prod_{i=n+2}^{\infty} X : \mathbf{f}_n \in S(\overline{\mathbf{f}})\}.$$

Proof. Let $\overline{x} \in \varprojlim F$. So, $x_i \in F(x_{i+1})$ for each $i \in \mathbb{N}$. The fact that $F(x) = \{f(x) : f \in \mathcal{F}\}$ for each $x \in X$, implies that for each $i \in \mathbb{N}$, $x_i = f_i(x_{i+1})$ for some $f_i \in \mathcal{F}$. Thus, $\overline{x} \in \varprojlim \overline{\mathbf{f}}$ for some $\overline{\mathbf{f}} \in S$.

On the other hand, let $\overline{\mathbf{f}} \in \mathcal{S}$ and let $\overline{x} \in \varprojlim \overline{\mathbf{f}}$. So, $x_i = f_i(x_{i+1})$ for each $i \in \mathbb{N}$. Hence, for each $i \in \mathbb{N}$, $x_i \in F(x_{i+1})$. This implies that $\overline{x} \in \varprojlim F$. $\square$

Observe that if $P \cap Dom(F) = \emptyset$, then $\varprojlim \overline{\mathbf{f}} \cap \varprojlim \overline{\mathbf{g}} = \emptyset$ for each $\overline{\mathbf{f}}, \overline{\mathbf{g}} \in \mathcal{S}$, $\overline{\mathbf{f}} \neq \overline{\mathbf{g}}$.

From now on, we will consider that $\mathcal{F}$ is contractive with module α and the function F satisfies that $Dom(F) \cap P = \emptyset$.

Lemma 4.6. *For each $\overline{\mathbf{f}} \in \mathcal{S}$, $\varprojlim \overline{\mathbf{f}}$ is a degenerate set.*

Proof. Let $\overline{x}, \overline{y} \in \varprojlim \overline{\mathbf{f}}$. We will prove that $\overline{x} = \overline{y}$ proving that for each $n \in \mathbb{N}$, $x_n = y_n$.

Let $n \in \mathbb{N}$. Since $\mathcal{F}$ is contractive, then we have that $d(x_n, y_n) \leq \alpha^m d(x_{n+m}, y_{n+m}) \leq \alpha^m$ for each $m \in \mathbb{N}$. The fact that $\alpha < 1$, implies that α^m converges to zero when $m \to \infty$. So, $d(x_n, y_n) = 0$. Therefore $x_n = y_n$. $\square$

Theorem 4.7. *$\varprojlim F$ is a totally disconnected space.*

Proof. Since $Dom(F) \cap P = \emptyset$, then $G'_n = \bigcup_{\mathbf{f} \in \mathcal{S}_n} G'_n(\mathbf{f})$ is the union of pairwise disjoint sets. Moreover, $G'_n(\mathbf{f})$ is connected for each $\mathbf{f} \in \mathcal{S}_n$, $n \in \mathbb{N}$. Let D be a component of $\varprojlim F$ and let $\overline{x}, \overline{y} \in D$ be such that $\overline{x} \neq \overline{y}$. By Proposition 4.5 and Lemma 4.6, there exist $\overline{\mathbf{f}}, \overline{\mathbf{g}} \in \mathcal{S}$ such that $\{\overline{x}\} = \varprojlim \overline{\mathbf{f}}$ and $\{\overline{y}\} = \varprojlim \overline{\mathbf{g}}$. The fact that $\overline{x} \neq \overline{y}$ implies that $\overline{\mathbf{f}} \neq \overline{\mathbf{g}}$. Then there exists $n \in \mathbb{N}$ such that $\mathbf{f}_n \neq \mathbf{g}_n$. So, $G'_n(\mathbf{f}_n) \cap G'_n(\mathbf{g}_n) = \emptyset$. Therefore $\overline{x}_{n+1}$ and $\overline{y}_{n+1}$ are in different components of G'_n. This implies that the points $\overline{x}$ and $\overline{y}$ are in different components, a contradiction. $\square$

Theorem 4.8. *$\varprojlim F$ is a perfect space.*

Proof. Let $\overline{x} \in \varprojlim F$. By Lemma 4.6, there exists $\overline{\mathbf{f}} \in \mathcal{S}$ such that $\{\overline{x}\} = \varprojlim \overline{\mathbf{f}}$. Moreover, from Proposition 4.5, $\varprojlim \overline{\mathbf{f}} = \bigcap_{\mathbf{f}_n \in S(\overline{\mathbf{f}})} \left(G'_n(\mathbf{f}_n) \times \prod_{i=n+2}^{\infty} X \right)$. For each $m \in \mathbb{N}$, let $\mathbf{f}^m = \{f_{m,i}\}_{i=1}^{\infty} \in \mathcal{S}$ where $f_{m,i} = f_i$ if $1 \leq i \leq m$ and $f_{m,m+1} \neq f_{m+1}$. For each $m \in \mathbb{N}$, let $\{\overline{x}^m\} = \varprojlim \mathbf{f}^m$. Since $Dom(F) \cap P = \emptyset$, $\overline{x}^k \neq \overline{x}^m$ for every $k, m \in \mathbb{N}$, $k \neq m$. Note that for each $m \in \mathbb{N}$, $\overline{x}^m \in G'_m(\mathbf{f}_m) \times \prod_{i=m+2}^{\infty} X$. Under this construction we have that the sequence $\{\overline{x}^m\}_{m=1}^{\infty}$ converges to $\overline{x}$. Therefore, $\varprojlim F$ is a perfect space. $\square$

The following result is a consequence of [6, Theorem 111], Theorem 4.7 and Theorem 4.8.

Corollary 4.9. *$\varprojlim F$ is homeomorphic to the Cantor set.*

In Corollary 4.9, F may or may not have a surjective graph. However, if F is a surjective function, then the condition $Dom(F) \cap P = \emptyset$ implies that for each $f, g \in \mathcal{F}$, $f \neq g$, $P_{fg} = \emptyset$. Hence, Examples 3.3, 3.4, 3.6 and 3.7 are particular cases of Corollary 4.9. Additionally, Example 3.5 is a particular case of the following Corollary.

Corollary 4.10. *If each $f \in \mathcal{F}$ satisfies that $f|_{Dom(F)}$ is a contractive mapping and $Dom(F) \cap P = \emptyset$, then $\varprojlim F$ is homeomorphic to the Cantor set.*

It is important to say that there exists $\mathcal{F}$ an uncountable family of contractive mappings such that the function F is the set theoretic union of the elements of $\mathcal{F}$, $P \cap Dom(F) = \emptyset$, and the inverse limit of F is the Hilbert Cube. Indeed, for each $x \in [0,1]$, let $f_x : [0,1] \to [0,1]$ defined by $f_x(t) = x$, $\mathcal{F} = \{f_x : x \in [0,1]\}$ and $F : [0,1] \to 2^{[0,1]}$ given by $F(t) = \{f_x(t) : f_x \in \mathcal{F}, x \in [0,1]\} = [0,1]$. Then the collection $\mathcal{F}$ and the function F satisfy the conditions. Moreover, it is not difficult to give an example where $\mathcal{F}$ is an uncountable family of contractive mappings such that the function F defined by the set theoretic union of the elements of $\mathcal{F}$ is upper semi-continuous, the graph of F is not connected, $P \cap Dom(F) = \emptyset$ and $\varprojlim F$ is not a Cantor set.

Finally, in the following example, $\varprojlim F$ is homeomorphic to the Cantor set, but F is not the union of a family of mappings from $[0,1]$ into itself.

Example 4.11. Let $F : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function defined by

$$F(x) = \begin{cases} \{0, \frac{1}{4}\} & \text{if } x \in [0, \frac{1}{2}], \\ \{\frac{3}{4}, 1\} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then, $\varprojlim F$ is homeomorphic to the Cantor set

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