

# Making holes in the hyperspace of subcontinua of a continuum having property (b)



José G. Anaya, Rosa I. Carranza, David Maya <sup>\*</sup>, Fernando Orozco-Zitli

Universidad Autónoma del Estado de México, Facultad de Ciencias, Instituto Literario 100, Col. Centro, Toluca, CP 50000, Mexico

## ARTICLE INFO

### Article history:

Received 22 April 2020  
Received in revised form 28 December 2020  
Accepted 29 December 2020  
Available online 5 January 2021

### MSC:

54B20  
54F55

### Keywords:

Continuum  
Hyperspace  
Property (b)  
Unicoherence

## ABSTRACT

Let  $C(X)$  be the hyperspace of all subcontinua of a metric continuum  $X$ . An element  $A \in C(X)$  makes a hole in  $C(X)$  if  $C(X) - \{A\}$  is not unicoherent. In this paper, we characterize the elements  $A \in C(X)$  satisfying that  $A$  makes a hole in  $C(X)$  when  $X$  has property (b).

© 2021 Elsevier B.V. All rights reserved.

## 1. Introduction

In Hyperspace Theory, it is often useful to know more about special kinds of points in the hyperspace of all subcontinua of a continuum. An example is the recent result stating that the simple closed curve is the unique continuum whose hyperspace of non-blocker closed subsets of its singletons is the hyperspace of singletons (see [8, Theorem 5.3, p. 78]).

Several authors have investigated various properties of special sets in continua. For instance, the authors in [8] and [10] provide a survey of results relating the notions of non-weak cut set, non-blocker set of singletons, shore set and non-cut set in a continuum.

Our aim is to study making holes property of an element in hyperspaces. In the class of continua having property (b), each subcontinuum that makes a hole in the hyperspace of all subcontinua is a cut set (see

<sup>\*</sup> Corresponding author.

E-mail addresses: [jgao@uaemex.mx](mailto:jgao@uaemex.mx) (J.G. Anaya), [rcarranzac@uaemex.mx](mailto:rcarranzac@uaemex.mx) (R.I. Carranza), [dmayae@outlook.com](mailto:dmayae@outlook.com), [dmayae@uaemex.mx](mailto:dmayae@uaemex.mx) (D. Maya), [forozcozitli@gmail.com](mailto:forozcozitli@gmail.com) (F. Orozco-Zitli).

Theorem 3.5). This kind of set has been studied in [1–7]. The problem that will be partially solved in this paper is presented below.

A connected topological space  $Z$  is *unicoherent* provided that  $A \cap B$  is connected whenever  $A$  and  $B$  are connected closed subsets of  $Z$  such that  $Z = A \cup B$ . An element  $z$  of a unicoherent topological space  $Z$  makes a hole in  $Z$  if  $Z - \{z\}$  is not unicoherent.

A *continuum* is a non-empty compact connected metric space. A *hyperspace* of a continuum  $X$  is a specified collection of non-empty closed subsets of  $X$  topologized with the Hausdorff metric (see [12, Definition 2.1, p. 11]). Given a continuum  $X$ ,  $C(X)$  denotes the hyperspace of all subcontinua of  $X$ . It is known that  $C(X)$  is unicoherent for every continuum  $X$  (see [12, Theorem 19.8, p. 159]).

In this paper we are interested in the following problem which arises in [1, p. 2000]:

**Problem.** Let  $\mathcal{H}(X)$  be a unicoherent hyperspace of  $X$ . For which elements  $A \in \mathcal{H}(X)$ ,  $A$  makes a hole in  $\mathcal{H}(X)$ .

In the current paper, we present the partial solution to this problem when  $\mathcal{H}(X) = C(X)$  and  $X$  has property (b).

## 2. Auxiliary results

The symbols  $\mathbb{N}$  and  $\mathbb{R}$  denote the set of all positive integers and the set of all real numbers, respectively.

An *arc* is any space homeomorphic to the closed unit interval  $[0, 1]$ . The word *map* stands for a continuous function between topological spaces.

A subspace  $Y$  of a topological space  $Z$  is a *deformation retract* of  $Z$  if there exists a map  $H : Z \times [0, 1] \rightarrow Z$  such that  $H(z, 0) = z$  for each  $z \in Z$ ,  $H(Z \times \{1\}) = Y$  and  $H(y, 1) = y$  for each  $y \in Y$ . A topological space  $Z$  is *contractible* if there exists  $z \in Z$  in such a way that  $\{z\}$  is a deformation retract of  $Z$ .

Denote by  $\exp : \mathbb{R} \rightarrow S^1$  the map defined by  $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$ . A connected topological space  $Z$  has property (b) provided that for each map  $f$  from  $Z$  into the unit circumference in the Euclidean plane  $S^1$  there exists a map  $h : Z \rightarrow \mathbb{R}$  (called *lift of  $f$* ) such that  $f = \exp \circ h$ . Observe that to have property (b) is a topological property. Each metric space having property (b) is unicoherent (see [14, Theorem 7.3, p. 227]). The known results related to property (b) which will be used frequently are presented below.

**Proposition 2.1.** [1, Proposition 8, p. 2001] *Let  $Z$  be a connected topological space and let  $W$  and  $Y$  be non-empty connected closed subsets of  $Z$  such that  $Z = W \cup Y$ . If  $W$  and  $Y$  have property (b) and  $W \cap Y$  is connected, then  $Z$  has property (b).*

**Proposition 2.2.** [1, Proposition 9, p. 2001] *Let  $Z$  be a connected topological space and let  $Y$  be a deformation retract of  $Z$ . Then  $Z$  has property (b) if and only if  $Y$  has property (b).*

**Proposition 2.3.** [1, Proposition 6, p. 2001] *If  $Z$  is a connected topological space and  $f, g : Z \rightarrow \mathbb{R}$  are maps such that  $\exp \circ f = \exp \circ g$  and  $f(z) = g(z)$  for some  $z \in Z$ , then  $f = g$ .*

**Lemma 2.4.** *Let  $Z$  and  $Y$  be connected metric spaces having property (b), let  $A$  be a connected subset of  $Y$  and let  $f_Y : Y \rightarrow S^1$ ,  $f_Z : Z \rightarrow S^1$  and  $g : A \rightarrow Z$  be maps. If  $h_Z : Z \rightarrow \mathbb{R}$  is a lift of  $f_Z$  and  $f_Y|_A = f_Z \circ g$ , then there exists a lift  $h_Y : Y \rightarrow \mathbb{R}$  of  $f_Y$  such that  $h_Y|_A = h_Z \circ g$ .*

**Proof.** Fix  $a \in A$  and set  $t = (h_Z \circ g)(a)$ . Since  $Y$  has property (b) and  $\exp(t) = \exp \circ (h_Z \circ g)(a) = (f_Z \circ g)(a) = f_Y|_A(a) = f_Y(a)$ , there exists a lift of  $f_Y$   $h_Y : Y \rightarrow \mathbb{R}$  satisfying that  $h_Y(a) = t$ . Now, from the fact  $\exp \circ (h_Y|_A) = f_Y|_A = f_Z \circ g = \exp \circ h_Z \circ g$  and Proposition 2.3, it follows that  $h_Y|_A = h_Z \circ g$ .  $\square$

Let  $X$  and  $Y$  be continua. A map  $g : X \rightarrow Y$  is said to be *monotone* provided that  $g^{-1}(y)$  is connected for each  $y \in Y$ .

**Theorem 2.5.** *Monotone images of continua having property (b) have property (b).*

**Proof.** Let  $X$  be a continuum such that there exist a continuum  $P$  having property (b) and a monotone surjective map  $q : P \rightarrow X$ . In order to prove that  $X$  has property (b), let  $f : X \rightarrow S^1$  be a map.

Since  $P$  has property (b), there exists a lift  $g : P \rightarrow \mathbb{R}$  of  $f \circ q$ . Define  $h : X \rightarrow \mathbb{R}$  by  $h(x) = g(y)$  whenever  $x \in X$ ,  $y \in Y$  and  $q(y) = x$ . We will prove that  $h$  is a lift of  $f$ .

Let  $x \in X$ . Since  $q^{-1}(x)$  is connected and  $f \circ q = \exp \circ g$ ,  $g(q^{-1}(x))$  is a connected subset of  $\mathbb{R}$  contained in  $\exp^{-1}(f(x))$ . Thus,  $g(q^{-1}(x))$  is a singleton. This proves that  $h$  is well defined.

To show the continuity of  $h$ , let  $\{x_n\}_{n=1}^\infty$  be a sequence in  $X$  and let  $x \in X$  be such that  $\lim x_n = x$ . For each  $n \in \mathbb{N}$ , take  $y_n \in q^{-1}(x_n)$ . Since  $P$  is compact, we may assume that there exists  $y \in P$  such that  $\lim y_n = y$ . Then,  $\lim g(y_n) = g(y)$  and  $\lim q(y_n) = q(y)$ . Thus,  $q(y) = x$  and so,  $\lim h(x_n) = h(x)$ . Hence,  $h$  is a map.

Finally, for each  $x \in X$ , we can choose  $y \in q^{-1}(x)$  to conclude that  $f(x) = (f \circ q)(y) = (\exp \circ g)(y) = (\exp \circ h)(x)$ . This implies that  $h$  is a lift of  $f$ .

Therefore,  $X$  has property (b).  $\square$

Given a continuum  $X$ ,  $F_1(X)$  denotes the hyperspace of all singletons of  $X$ , this is  $F_1(X) = \{\{x\} : x \in X\}$ . Note that  $F_1(X)$  is a closed subset of  $C(X)$ .

A *Whitney map* for  $C(X)$  is a map  $\mu : C(X) \rightarrow [0, 1]$  such that:

1.  $\mu(\{x\}) = 0$  for each  $x \in X$ ,
2.  $\mu(A) < \mu(B)$  whenever  $A \subseteq B$  and  $A \neq B$ ,
3.  $\mu(X) = 1$ .

Whitney maps always exist for every continuum (see [12, Theorem 13.4, p. 107]). Throughout this paper,  $\mu$  will denote a Whitney map for  $C(X)$ .

Let  $X$  be a continuum. An *order arc* in  $C(X)$  is an arc  $\alpha$  in  $C(X)$  such that if  $A, B \in \alpha$ , then either  $A \subseteq B$  or  $B \subseteq A$ . If  $\alpha$  is an order arc in  $C(X)$ , then  $\alpha$  is said to be an *order arc from*  $\bigcap \alpha$  *to*  $\bigcup \alpha$ . Order arcs  $\alpha$  are homeomorphic to  $[0, 1]$  (see [12, Lemma 14.2, p. 111]) and the homeomorphism  $\beta_\alpha : [0, 1] \rightarrow C(X)$  can be chosen in such a way  $\mu(\beta_\alpha(t)) = (1 - t)\mu(\bigcap \alpha) + t\mu(\bigcup \alpha)$ .

### 3. Continua having property (b)

The main result of this section (Theorem 3.4) characterizes all subcontinua of  $X$  that make a hole in  $C(X)$  when  $X$  has property (b). Such partial solution to Problem is the aim of the current paper. In the following section, Theorem 3.4 is applied to solve Problem in some interesting subclasses of continua having property (b).

For sake of simplicity, throughout this section  $X$  will denote a continuum having property (b).

**Theorem 3.1.** *Let  $\mathcal{H}$  be a subset of  $C(X)$ . If  $F_1(X) \subseteq \mathcal{H}$  and  $C(A) \subseteq \mathcal{H}$  for each  $A \in \mathcal{H}$ , then  $\mathcal{H}$  has property (b).*

**Proof.** Let  $f : \mathcal{H} \rightarrow S^1$  be a map. Since  $F_1(X)$  is homeomorphic to  $X$ ,  $F_1(X)$  has property (b). Let  $h_1 : F_1(X) \rightarrow \mathbb{R}$  be a map such that  $f|_{F_1(X)} = \exp \circ h_1$ . On the other hand, for each  $A \in \mathcal{H}$ ,  $C(A)$  has property (b) (see [1, Lemma 13, p. 2004]) and so  $f|_{C(A)}$  has a lift. Now, let  $in : F_1(A) \rightarrow C(A)$  be the

inclusion map. Then  $(f|_{C(A)})|_{F_1(A)} = f|_{F_1(X)} \circ in$ . Applying Lemma 2.4, there exists a lift  $h_A : C(A) \rightarrow \mathbb{R}$  of  $f|_{C(A)}$  such  $h_A|_{F_1(A)} = h_1 \circ in$ . Define  $h : \mathcal{H} \rightarrow \mathbb{R}$  by  $h(A) = h_A(A)$ . We will prove that  $h$  is continuous.

Let  $\{A_n\}_{n \in \mathbb{N}}$  be a sequence converging to  $A_0$  in  $\mathcal{H}$ . Fix a sequence of points  $\{a_n\}_{n \in \mathbb{N}}$  in  $X$  converging to  $a_0 \in X$  such that  $a_n \in A_n$  for each  $n \in \mathbb{N} \cup \{0\}$  and a sequence of order arcs  $\{\alpha_n\}_{n \in \mathbb{N}}$  in  $C(X)$  such that  $\bigcap \alpha_n = \{a_n\}$  and  $\bigcup \alpha_n = A_n$  for each  $n \in \mathbb{N}$ . By the compactness of  $C(X)$ , we may assume that  $\{\alpha_n\}_{n \in \mathbb{N}}$  converges to an order arc  $\alpha_0$  from  $\{a_0\}$  to  $A_0$  in  $C(X)$ .

Next, set  $W = \{a_n : n \in \mathbb{N} \cup \{0\}\}$  and denote by  $Y$  the subspace  $(X \times \{0\}) \cup (W \times [0, 1])$  of  $X \times [0, 1]$ . Define  $H : Y \times [0, 1] \rightarrow Y$  by  $H((x, l), t) = (x, (1 - t)l)$  to get a map satisfying that  $H((x, l), 0) = (x, l)$  for each  $(x, l) \in Y$ ,  $H(Y \times \{1\}) = X \times \{0\}$  and  $H((x, 0), 1) = (x, 0)$  for each  $x \in X$ . Thus,  $X \times \{0\}$  is a deformation retract of  $Y$ . Since  $X$  is homeomorphic to  $X \times \{0\}$ , we conclude that  $X \times \{0\}$  has property (b) and so, by Proposition 2.2,  $Y$  has property (b). Now, define the map  $\pi : Y \rightarrow \mathcal{H}$  by

$$\pi((x, l)) = \begin{cases} \{x\}, & \text{if } l = 0, \\ \beta_{\alpha_n}(l), & \text{if } l > 0 \text{ and } x = a_n. \end{cases}$$

From the choice of  $\alpha_n$  and  $\beta_{\alpha_n}$ , it follows that  $\pi$  is a map. Therefore, the map  $f \circ \pi : Y \rightarrow S^1$  has a lift. On the other hand,  $X \times \{0\}$  is a connected closed subset of  $Y$  such that  $\pi(X \times \{0\}) = F_1(X)$ . Since  $h_1$  is a lift of  $f|_{F_1(X)}$  and  $(f \circ \pi)|_{X \times \{0\}} = f|_{F_1(X)} \circ (\pi|_{X \times \{0\}})$ , Lemma 2.4 is applied to obtain that there exists a lift of  $f \circ \pi|_{h_Y} : Y \rightarrow \mathbb{R}$  such that  $h_Y|_{X \times \{0\}}((x, 0)) = h_1 \circ \pi|_{X \times \{0\}}((x, 0))$  for all  $x \in X$ .

Let  $n \in \mathbb{N} \cup \{0\}$ . We will show that  $h_Y((a_n, 1)) = h(A_n)$ . To this end, observe that

$$\begin{aligned} \exp \circ (h_Y|_{\{a_n\} \times [0, 1]}) &= (\exp \circ h_Y)|_{\{a_n\} \times [0, 1]} \\ &= (f \circ \pi)|_{\{a_n\} \times [0, 1]} \\ &= f|_{C(A_n)} \circ \pi|_{\{a_n\} \times [0, 1]} \\ &= (\exp \circ h_{A_n}) \circ \pi|_{\{a_n\} \times [0, 1]} \\ &= \exp \circ ((h_{A_n} \circ \pi)|_{\{a_n\} \times [0, 1]}) \end{aligned}$$

and

$$\begin{aligned} (h_{A_n} \circ \pi|_{\{a_n\} \times [0, 1]})(a_n, 0) &= h_{A_n}(\pi|_{\{a_n\} \times [0, 1]}((a_n, 0))) \\ &= h_{A_n}(\{a_n\}) \\ &= h_1(\{a_n\}) \\ &= h_1 \circ \pi((a, 0)) \\ &= h_Y((a, 0)). \end{aligned}$$

By Proposition 2.3, we infer that  $h_Y|_{\{a_n\} \times [0, 1]} = (h_{A_n} \circ \pi)|_{\{a_n\} \times [0, 1]}$ . Thus,

$$\begin{aligned} h_Y((a_n, 1)) &= (h_{A_n} \circ \pi|_{\{a_n\} \times [0, 1]})(a_n, 1) \\ &= h_{A_n}(\pi|_{\{a_n\} \times [0, 1]}((a_n, 1))) \\ &= h_{A_n}(\beta_{\alpha_n}(1)) \\ &= h_{A_n}(A_n). \end{aligned}$$

Finally,  $\lim h(A_n) = \lim h_Y((a_n, 1)) = h_Y((a_0, 1)) = h(A_0)$ .

Therefore,  $h$  is a map satisfying  $f = \exp \circ h$ .  $\square$

The results below are immediate consequences of the last theorem.

**Corollary 3.2.** *Let  $A \in C(X) - F_1(X)$ . Then  $\mu^{-1}([0, \mu(A)]) - \{A\}$  has property (b).*

**Corollary 3.3.** *The element  $X$  does not make a hole in  $C(X)$ .*

Note that a consequence of [1, Theorem 3, p. 2001] and Corollary 3.3 is that if either  $A \in F_1(X)$  or  $A = X$ , then  $A$  does not make a hole in  $C(X)$ . So, to solve Problem, it suffices to characterize all non-degenerate proper subcontinua that make a hole in  $C(X)$ . The non-degenerate proper subcontinua are called *non-trivial*.

The next result gives the partial solution to Problem when  $X$  has property (b) by showing that the unicoherence of the space obtained from  $C(X)$  by removing one of its points  $A$  can be established either by the connectedness of the space obtained from a Whitney level containing  $A$  by removing  $A$ , or by certain topological behaviour of the space  $X - A$ .

Given a non-trivial subcontinuum  $A$  of  $X$ , the symbol  $X - A = U \parallel V$  means that  $U$  and  $V$  are non-empty disjoint open subsets of  $X$  such that  $X - A = U \cup V$  and each  $B \in \mu^{-1}(\mu(A))$  satisfies either  $B \subseteq U \cup A$  or  $B \subseteq V \cup A$ .

**Theorem 3.4.** *Let  $A$  be a non-trivial subcontinuum of  $X$ . Then the following statements are equivalent.*

- (1)  $A$  makes a hole in  $C(X)$ .
- (2)  $X - A = U \parallel V$ .
- (3)  $\mu^{-1}(\mu(A)) - \{A\}$  is not connected.

**Proof.** The equivalence between (2) and (3) is proved in [11, Theorem 2.1, p. 210]. Let us prove that (1) and (3) are equivalent.

Now, set  $t = \mu(A)$ ,  $W = \mu^{-1}([0, t]) - \{A\}$  and  $Y = \mu^{-1}([t, 1]) - \{A\}$ . Observe that  $W$  and  $Y$  are connected closed subsets of  $C(X) - \{A\}$ ,  $C(X) - \{A\} = W \cup Y$  and  $W \cap Y = \mu^{-1}(t) - \{A\}$ . By Corollary 3.2,  $W$  has property (b). Since  $Y$  satisfies the conditions of [1, Lemma 13, p. 2004],  $Y$  has property (b).

Next, assume that  $A$  makes a hole in  $C(X)$ . By Proposition 2.1, if  $W \cap Y = \mu^{-1}(t) - \{A\}$  were connected, then  $C(X) - \{A\} = W \cup Y$  would have property (b) and so  $C(X) - \{A\}$  would be unicoherent. Hence,  $\mu^{-1}(t) - \{A\}$  is not connected. This proves that (1) implies (3).

Finally, if  $\mu^{-1}(t) - \{A\}$  is not connected, then  $C(X) - \{A\}$  is the union of the connected closed subsets  $W$  and  $Y$  of  $C(X) - \{A\}$  whose intersection is not connected. This implies that  $C(X) - \{A\}$  is not unicoherent. In conclusion,  $A$  makes a hole in  $C(X)$ . Thus, (1) is implied by (3).  $\square$

The condition  $X - A = U \parallel V$  implies that  $X - A$  must not be connected. So, the next result follows from Theorem 3.4. This condition becomes a first criterion to determine if a subcontinuum makes a hole in  $C(X)$ .

**Corollary 3.5.** *Let  $A$  be a non-trivial subcontinuum of  $X$ . If  $A$  makes a hole in  $C(X)$ , then  $X - A$  is not connected.*

Theorem 5.1 shows a class of continua for which the converse of last corollary is true.

A continuum is said to be *decomposable* provided that it can be written as the union of two proper subcontinua. A continuum which is not decomposable is said to be *indecomposable*.

The proof of the next result follows from Corollary 3.5 and the fact that if  $X$  is indecomposable, then  $X - A$  is connected for all non-trivial subcontinuum  $A$  of  $X$ .

**Theorem 3.6.** *If  $X$  is an indecomposable continuum having property (b), then  $A$  does not make a hole in  $C(X)$  for any  $A \in C(X)$ .*

The next example proves that the converse of last result does not hold.

**Example 3.7.** Let  $X$  be the Cantor fan. Then  $X$  is decomposable and since  $X$  is contractible, by Proposition 2.2, we have that  $X$  has property (b). Now, by [4, Theorem 3.8, p. 136] and the fact that the interior of each arc contained in  $X$  is empty, if  $A \in C(X)$ , then  $A$  does not make a hole in  $C(X)$ .

#### 4. Arcwise connected continua

A continuum  $X$  is said to be *arcwise connected* provided that any two points of  $X$  can be joined by an arc in  $X$ . Applying Theorem 3.4 in the class of arcwise connected continuum having property (b), it can be proved that each subcontinuum that makes a hole in  $C(X)$  must be an arc having additionally properties.

In this section,  $X$  will denote an arcwise connected continuum having property (b) and  $A$  will be a non-trivial element of  $C(X)$ . Recall that a point  $x \in X$  is an *end point* of  $X$  if  $L - \{x\}$  is connected for any arc  $L$  in  $X$  containing it. The symbol  $E(X)$  will denote the set of all end points of  $X$ .

**Lemma 4.1.** *If  $X - A = U \parallel V$  and  $h : [0, 1] \rightarrow X$  is an one-to-one map such that  $h(0) \in U$  and  $h(1) \in V$ , then there exist  $s, r \in (0, 1)$  satisfying that  $A = h([s, r])$ ,  $h([0, s]) \subseteq U$  and  $h((r, 1]) \subseteq V$ .*

**Proof.** Set  $r = \inf h^{-1}(V)$ . Since  $V$  is an open subset of  $X$ ,  $r \in [0, 1] - h^{-1}(V)$ . Now, our assumption  $X = U \cup A \cup V$  and  $h(r) \in X - V$  imply that  $h(r) \in A \cup U$ . If  $r$  were an element of the open subset  $h^{-1}(U)$  of  $[0, 1]$ , then the set  $h^{-1}(U) \cap h^{-1}(V)$  would be non-empty, a contradiction to the fact that  $U \cap V = \emptyset$ . Thus,  $h(r) \in A$ .

Next, define  $S = \{t \in [0, r] : h([t, r]) \subseteq A\}$  and set  $s = \inf S$ . Then,  $h([s, r]) \subseteq A$  and  $0 < s \leq r$ . Now, suppose that  $h([s, r]) \subsetneq A$ . Fix  $p \in (r, 1]$  such that  $\mu(h([s, p])) < \mu(A)$ . Let  $\alpha$  be an order arc in  $C(X)$  such that  $\bigcap \alpha = h([0, p])$  and  $\bigcup \alpha = X$ . Define  $g : [0, 1] \rightarrow C(X)$  by

$$g(x) = \begin{cases} h([s - x, p]) & \text{if } x \in [0, s], \\ \beta_\alpha((x - s)(1 - s)^{-1}) & \text{if } x \in [s, 1] \end{cases}$$

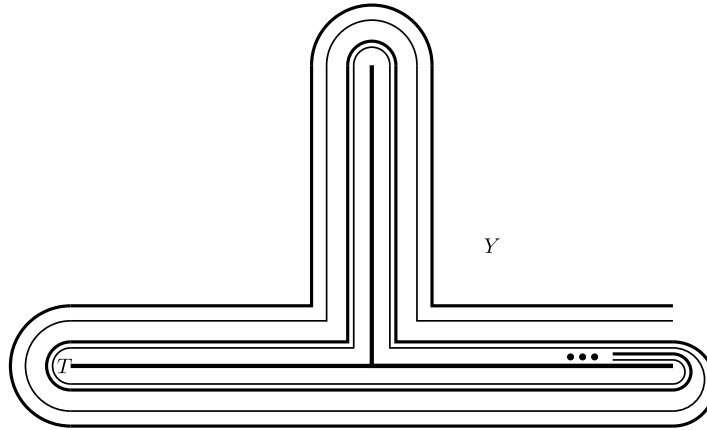
to get a map satisfying that  $g(0) = h([s, p])$ ,  $g(1) = X$  and  $g(0) \subseteq g(w) \subseteq g(1)$  for all  $w \in [0, 1]$ . Thus, there exists  $y \in (0, 1)$  such that  $g(y) \in \mu^{-1}(\mu(A))$ . Since  $h(p) \in g(0) \subseteq g(y)$  and  $h(p) \in V \subseteq X \setminus A$ , we have that  $g(y) \neq A$ . Let us prove that  $g(y) \cap U \neq \emptyset$  and  $g(y) \cap V \neq \emptyset$ . First, from fact that  $p > r$ , it follows that  $h((r, p)) \cap V \neq \emptyset$ . Hence,  $h([s, p]) \cap V \neq \emptyset$ . This and the inclusion  $h([s, p]) \subseteq g(y)$  imply that  $g(y) \cap V \neq \emptyset$ . Next, since  $y > 0$ , there exists  $l \in [0, s)$  such that  $h([l, p]) \subseteq g(y)$ . By the choice of  $s$ ,  $h([l, s]) \cap (X - A) \neq \emptyset$ . From this and the fact that  $X - A = U \cup V$ , it follows that  $(h([l, s]) \cap U) \cup (h([l, s]) \cap V) \neq \emptyset$ . The inequality  $s \leq r$  and the choice of  $r$  guarantee that  $h([l, s]) \cap V = \emptyset$ . Thus,  $h([l, s]) \cap U \neq \emptyset$ , and so  $g(y) \cap U \neq \emptyset$ . Hence,  $g(y) \cap U \neq \emptyset$  and  $g(y) \cap V \neq \emptyset$ . This contradicts the assumption  $X - A = U \parallel V$ . In conclusion,  $h([s, r]) = A$ .

To complete this proof, notice that  $h([0, s])$ ,  $h((r, 1])$  are connected subsets of  $X - A$  such that  $h([0, s]) \cap U \neq \emptyset$  and  $h((r, 1]) \cap V \neq \emptyset$ . Then  $h([0, s]) \subseteq U$  and  $h((r, 1]) \subseteq V$ .  $\square$

A point  $b \in A$  is called *arcwise accessible from  $X - A$*  provided that there exists an arc in  $(X - A) \cup \{b\}$  having  $b$  as an end point. The set of all arcwise accessible points from  $X - A$  is denoted by  $P(A)$ .

**Theorem 4.2.** *If  $A$  makes a hole in  $C(X)$ , then  $A$  is an arc and  $P(A) = E(A)$ .*

**Proof.** From Theorem 3.4, it follows that  $X - A = U \parallel V$ . Fix  $x \in U$  and  $y \in V$ . Since  $X$  is arcwise connected, there exists an one-to-one map  $h : [0, 1] \rightarrow X$  such that  $h(0) = x$  and  $h(1) = y$ . By Lemma 4.1, there exist  $s, r \in (0, 1)$  such that  $A = h([s, r])$ ,  $h([0, s]) \subseteq U$  and  $h((r, 1]) \subseteq V$ . Hence,  $A$  is an arc whose end points are  $h(s)$  and  $h(r)$ . Moreover,  $h(s)$  is an end point of the arc  $h([0, s]) \subseteq (X - A) \cup \{h(s)\}$  and the arc  $h([r, 1])$  has  $h(r)$  as end point and is a subset of  $(X - A) \cup \{h(r)\}$ . This proves that  $E(A) \subseteq P(A)$ .



**Fig. 1.** Continuum  $Y$ .

Now, let  $z \in P(A)$ . Then there exists an arc  $I$  in  $(X - A) \cup \{z\}$  having  $z$  as end point. Since  $I - \{z\}$  is connected subset of  $X - A$ , either  $I - \{z\} \subseteq U$  or  $I - \{z\} \subseteq V$ . Suppose that  $I - \{z\} \subseteq U$ . Let  $w \in E(I) - \{z\}$  and  $v \in V$ . Then there exists an one-to-one map  $g : [0, 1] \rightarrow X$  such that  $g(0) = w$ ,  $g(1) = v$  and  $I \subseteq g([0, 1])$ . Apply Lemma 4.1 to get  $p, q \in (0, 1)$  such that  $A = g([p, q])$ ,  $g([0, p]) \subseteq U$  and  $g((q, 1]) \subseteq V$ . Note that  $I - \{z\} \subseteq g([0, p])$  and  $z \in g([p, q])$ . Then  $z = g(p)$  and this implies that  $z \in E(A)$ . Therefore,  $P(A) = E(A)$ .  $\square$

The conclusion in Theorem 4.2 does not hold without the hypothesis of arcwise connectedness on  $X$ . We shall illustrate it with the next example.

**Example 4.3.** There exist a continuum  $Y$  and  $T \in C(Y)$  such that  $T$  makes a hole in  $C(Y)$  and  $T$  is not an arc.

Let  $Y = R_1 \cup T \cup R_2$  be the simplest compactification of two half-lines  $R_1$  and  $R_2$  with a simple triod  $T$  as remainder (see Fig. 1).

Then,  $Y$  is not arcwise connected. We will see that  $Y$  has property (b).

Note that, for each  $i \in \{1, 2\}$ ,  $R_i \cup T$  is a monotone image of the continuum  $P = \{x : |x| \leq 1\} \cup \{(t + 2)(t + 1)^{-1}e^{it} : t \in [0, \infty)\}$  contained in the complex plane obtained from the union of the unit complex disk and a spiral approaching to the unit complex circle. By [9, Example 12 and Theorem 54, pp. 61 and 68],  $P$  has property (b). Applying Theorem 2.5, we conclude that  $R_1 \cup T$  and  $R_2 \cup T$  have property (b). Finally, invoke Proposition 2.1 to conclude that  $Y$  has property (b).

Now, let us prove that  $T$  makes a hole in  $C(Y)$ . Observe that  $R_1$  and  $R_2$  are non-empty disjoint open subsets of  $Y$  such that  $Y - T = R_1 \cup R_2$  and if  $B \in C(Y)$  satisfying that  $B \cap R_1 \neq \emptyset$  and  $B \cap R_2 \neq \emptyset$ , then  $T \subsetneq B$  and this implies that  $\mu(T) < \mu(B)$ . Thus,  $Y - T = R_1 \parallel R_2$ . Therefore, by Theorem 3.4,  $T$  makes hole in  $C(Y)$ .

If  $A$  is an arc and  $a \in E(A)$ , define  $L(a)$  as follows:  $x \in L(a)$  if and only if  $x \in X - A$  and for all arc  $I$  contained in  $X$  such that  $E(I) = \{a, x\}$ , the inclusion  $A \subseteq I$  holds.

**Theorem 4.4.** *If  $A$  makes a hole in  $C(X)$ , then  $L(a)$  is a non-empty open subset of  $X$  for each  $a \in E(A)$ .*

**Proof.** Set  $a \in E(A)$ . Now, by Theorem 3.4,  $X - A = U \parallel V$ . Let us prove that either  $L(a) = U$  or  $L(a) = V$ .

First, let  $x \in U$  and  $y \in V$ . Choose an one-to-one map  $h : [0, 1] \rightarrow X$  such that  $h(0) = x$  and  $h(1) = y$ . Apply Lemma 4.1 to get  $s, r \in (0, 1)$  satisfying that  $A = h([s, r])$ ,  $h([0, s]) \subseteq U$  and  $h((r, 1]) \subseteq V$ . Since

$a \in E(A)$ , either  $h(s) = a$  or  $h(r) = a$ . Suppose that  $h(r) = a$ . Under this assumption, the inclusion  $x \in L(a)$  will be proved. Let  $I$  be an arc such that  $E(I) = \{x, a\}$ . Let  $g : [0, 1] \rightarrow X$  be an one-to-one map such that  $g(0) = x$ ,  $g(1) = a$  and  $I \cup h([r, 1]) = g([0, 1])$ . Lemma 4.1 ensures that there exists  $p, q \in (0, 1)$  such that  $A = g([p, q])$ ,  $g([0, p]) \subseteq U$  and  $g((q, 1]) \subseteq V$ . Thus,  $a = g(q)$ . Then  $g([0, q]) = I$  and  $A \subseteq I$ . In conclusion,  $x \in L(a)$ . Moreover,  $U \subseteq L(a)$ . Observe that the inclusion  $g((q, 1]) \subseteq V$  implies that  $V \cap L(a) = \emptyset$ . Hence  $L(a) \subseteq U$ . Therefore,  $L(a) = U$ , and so  $L(a)$  is a non-empty open subset of  $X$ .  $\square$

An arc  $A$  is called *simple arc* if  $P(A) = E(A) = \{a, b\}$  and  $L(a), L(b)$  are non-empty disjoint open subsets of  $X$  such that  $X - A = L(a) \cup L(b)$ .

The result below is immediate consequence of Theorems 4.2 and 4.4.

**Theorem 4.5.** *If  $A$  makes a hole in  $C(X)$ , then  $A$  is a simple arc.*

The next example verifies that the converse of this last result fails.

A continuum is *hereditarily indecomposable* if each one of its subcontinua is indecomposable. Note that every hereditarily indecomposable continuum contains no arc.

**Example 4.6.** There exists an arcwise connected continuum  $X$  containing a simple arc  $A$  such that  $A$  does not make a hole in  $C(X)$ .

Let  $TC(P)$  be the topological cone over a hereditarily indecomposable continuum  $P$  (see [13, 3.15, p. 41]), let  $\pi : P \times [0, 1] \rightarrow TC(P)$  be the natural map and fix  $p \in P$ . Denote by  $X$  the subspace  $\pi(\{(x, t) : x \in P, 2^{-1} \leq t \leq 1\} \cup \{(p, t) : 0 \leq t \leq 2^{-1}\})$  of  $TC(P)$ . Note that the continuum  $X$  is contractible. Hence, we infer that  $X$  is arcwise connected and, using Proposition 2.2,  $X$  has property (b).

Now,  $A = \pi(\{(p, t) : 3^{-1} \leq t \leq 2(3^{-1})\})$  is an arc with end points  $a = \pi((p, 3^{-1}))$  and  $b = \pi((p, 2(3^{-1}))$ ). Since  $P$  contains no arc,  $X$  is uniquely arcwise connected. Then,  $P(A) = E(A)$ ,  $L(b) = \pi(\{(p, t) : 0 \leq t < 3^{-1}\})$  and  $L(a) = X - \pi(\{(p, t) : 0 \leq t \leq 2(3^{-1})\})$ . Thus,  $L(a)$  and  $L(b)$  are non-empty connected open subsets of  $X$  such that  $X - A = L(a) \cup L(b)$ . Therefore,  $A$  is a simple arc.

We will use Theorem 3.4 to prove that  $A$  does not make a hole in  $C(X)$ . Let  $U$  and  $V$  be non-empty open subsets of  $X$  such that  $X - A = U \cup V$ . By the connectedness of  $L(a)$  and of  $L(b)$ , we have either  $U = L(a)$  and  $V = L(b)$  or  $U = L(b)$  and  $V = L(a)$ . Let us show that there exists  $B \in \mu^{-1}(\mu(A))$  such that  $B \cap U \neq \emptyset$  and  $B \cap V \neq \emptyset$  to conclude that  $X - A \neq U \cup V$ . Let  $\alpha$  be an order arc in  $C(X)$  such that  $\bigcap \alpha = \{\pi((p, 2^{-1}))\}$  and  $\bigcup \alpha = \pi(\{(x, 2^{-1}) : x \in P\})$ . Define  $g : [0, 1] \rightarrow C(X)$  by  $g(t) = \pi(\{p\} \times [(1-t)(3^{-1}), (1+t)(2^{-1})]) \cup \beta_\alpha(t)$  to get a map satisfying that  $g(0) = \pi(\{p\} \times [3^{-1}, 2^{-1}])$ ,  $g(1) = \pi(\{p\} \times [0, 1]) \cup \pi(\{(x, 2^{-1}) : x \in P\})$ ,  $g(0) \subsetneq A \subsetneq g(1)$  and  $g(0) \subsetneq g(w) \subsetneq g(1)$  for all  $w \in (0, 1)$ . By the continuity of  $\mu \circ g$ , there exists  $s \in (0, 1)$  such that  $g(s) \in \mu^{-1}(\mu(A))$ . Thus,  $g(s) \in C(X)$  is such that  $\mu(g(s)) = \mu(A)$ ,  $g(s) \cap U \neq \emptyset$  and  $g(s) \cap V \neq \emptyset$ . Therefore,  $X - A \neq U \cup V$ . By Theorem 3.4,  $A$  does not make a hole in  $C(X)$ .

Naturally, one can ask if each simple arc makes a hole in the hyperspace of subcontinua of a hereditarily arcwise connected continuum. In the next subsection, we show that this statement is true for dendroids.

#### 4.1. Dendroids

A *dendroid* is an arcwise connected, hereditarily unicoherent continuum (*hereditarily unicoherent* means that each subcontinuum is unicoherent). Dendroids have property (b) (see [13, 12.69, p. 271]). So, Theorem 4.5 can be applied to classify all subcontinua making a hole in  $C(X)$  when  $X$  is a dendroid.

**Theorem 4.7.** *Let  $X$  be a dendroid and let  $A \in C(X)$ . The subcontinuum  $A$  makes a hole in  $C(X)$  if and only if  $A$  is a simple arc.*

**Proof.** First, assume that  $A$  is an simple arc. Let  $a$  and  $b$  be the end points of  $A$ . In light of Theorem 3.4, it suffices to prove that  $X - A = L(a) \parallel L(b)$ . From our assumption,  $L(a)$  and  $L(b)$  are non-empty disjoint open subset of  $X$  such that  $X - A = L(a) \cup L(b)$ . Now, let  $B \in \mu^{-1}(\mu(A))$ . Suppose that there exist  $x \in B \cap L(a)$  and  $y \in B \cap L(b)$ . Let  $J$  and  $K$  be the unique arcs in  $X$  such that  $E(J) = \{a, x\}$  and  $E(K) = \{b, y\}$ . Thus,  $A \subseteq J$ ,  $A \subseteq K$  and  $E(J) \cap E(K) = \emptyset$ . Then,  $J \cup K$  must be the unique arc in  $X$  such that  $E(J \cup K) = \{x, y\}$ . Set  $I = J \cup K$ . Since  $X$  is hereditarily unicoherent,  $I \cap B$  is a subcontinuum of  $X$  contained in  $I$  containing  $x$  and  $y$ . Hence,  $I \cap B = I$ . This implies that  $I \subseteq B$  and so,  $A$  is a proper subset of  $B$ . Therefore  $\mu(A) < \mu(B)$ , a contradiction. In conclusion, either  $B \subseteq A \cup L(a)$  or  $B \subseteq A \cup L(b)$ .

The necessity follows from Theorem 4.5.  $\square$

### 5. Chainable continua

In this section, exploring for which class of continua the converse of Corollary 3.5 is true, we show that it holds for chainable continua while it does not for irreducible continua.

A *chain* in a continuum  $X$  is a non-empty, finite, indexed collection  $\mathcal{C} = \{U_1, U_2, \dots, U_n\}$  of open subsets of  $X$  such that  $U_i \cap U_j \neq \emptyset$  if and only if  $|i - j| \leq 1$ . If  $\mathcal{C}$  is a chain in  $X$  such that  $\max\{\text{diam}(U_i) : 1 \leq i \leq n\} < \varepsilon$ , then  $\mathcal{C}$  is called  $\varepsilon$ -*chain* in  $X$ . A continuum  $X$  is *chainable* provided that there is an  $\varepsilon$ -chain in  $X$  covering  $X$  for each  $\varepsilon > 0$ . Chainable continua have property (b) (see [13, Theorem 12.11, p. 235] and [13, 11.6, p. 269]). Then, Theorem 3.4 can be used to characterize the subcontinua  $A$  of a chainable continuum  $X$  such that makes hole in  $C(X)$ .

A continuum is said to be *irreducible* provided that there exist  $p, q \in X$  such that no proper subcontinuum of  $X$  contains  $\{p, q\}$ .

We know that every chainable continuum is irreducible and, in fact, hereditarily irreducible (see [13, Theorems 12.5 and 12.11, pp. 233 and 235]).

**Theorem 5.1.** *Let  $X$  be a chainable continuum and let  $A$  be a non-trivial subcontinuum of  $X$ . Then  $A$  makes a hole in  $C(X)$  if and only if  $X - A$  is not connected.*

**Proof.** The necessity follows from Corollary 3.5. It only remains to prove the sufficiency.

Our assumption on  $X$  implies that  $X$  is irreducible and so there exist  $p, q \in X$  such that no proper subcontinuum of  $X$  contains  $\{p, q\}$ . Since  $X - A$  is not connected, by [13, Theorem 11.7, p. 199] there exist components  $C$  and  $D$  of  $X - A$ , each one of them is open in  $X$ , one containing  $p$  and the other containing  $q$ , and  $X - A = C \cup D$ . We will prove that  $X - A = C \parallel D$ . To this end, let  $B \in C(X)$  be such that  $B \cap C \neq \emptyset$  and  $B \cap D \neq \emptyset$  and let  $a \in A$ . We shall prove that  $a$  is an element of the closure of  $B$ . Let  $\varepsilon > 0$ . Fix  $x \in B \cap C$  and  $y \in B \cap D$ , and choose  $\delta > 0$  satisfying that  $\delta < \varepsilon$ ,  $\delta < \min\{d(x, b) : b \in A \cup D\}$ ,  $\delta < \min\{d(y, b) : b \in A \cup C\}$ ,  $3\delta < d(x, y)$  and  $3\delta < \sup\{d(b, d) : b, d \in A\}$ .

Let  $\mathcal{C} = \{U_1, U_2, \dots, U_n\}$  be a  $\delta$ -chain in  $X$  covering  $X$ . Let  $j, k \in \{1, \dots, n\}$  be such that  $x \in U_j$  and  $y \in U_k$ . The condition  $\delta < d(x, y)$  and the fact that  $\mathcal{C}$  is a chain together imply that  $|j - k| > 1$ . We may assume that  $j < k$ .

Now, from the inequality  $\delta < \min\{d(x, b) : b \in A \cup D\}$ , it follows that  $U_j \cap (A \cup D) = \emptyset$ . Then  $A \cup D$  is contained in the union of the disjoint non-empty open subsets  $\bigcup_{i=1}^{j-1} U_i$  and  $\bigcup_{i=j+2}^n U_i$ . By [13, Corollary 5.9, p. 75],  $A \cup D$  is connected. So, since  $j < k$ , we conclude that  $A \cup D \subseteq \bigcup_{i=j+2}^n U_i$  and  $(A \cup D) \cap \bigcup_{i=1}^{j-1} U_i = \emptyset$ . Similar arguments prove that  $(A \cup C) \cap \bigcup_{i=j+2}^n U_i = \emptyset$ .

Set  $\zeta = \min\{i \in \{1, \dots, n\} : U_i \cap (A \cup D) \neq \emptyset\}$  and  $\kappa = \max\{i \in \{1, \dots, n\} : U_i \cap (A \cup C) \neq \emptyset\}$ . Observe that  $j < \zeta$  and  $\kappa < k$ . Let  $c, e \in A$  be such that  $3\delta < d(c, e)$  and choose  $l, m \in \{1, \dots, n\}$  in such a way  $c \in U_l$  and  $e \in U_m$ . Since  $\mathcal{C}$  is a chain, the inequality  $|l - m| > 1$  holds. For the choice of  $\zeta$  and  $\kappa$ , we have that  $\zeta \leq \min\{l, m\} < \max\{l, m\} \leq \kappa$ . So,  $j < \zeta < \kappa < k$ .

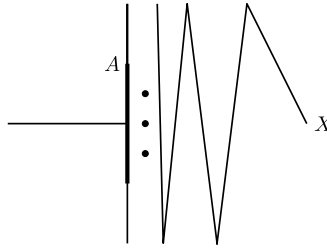


Fig. 2. Continuum  $X$ .

Finally, let  $r \in \{1, \dots, n\}$  be such that  $a \in U_r$ . Since  $a \in A \cup D$  and  $a \in A \cup C$ ,  $\zeta \leq r \leq \kappa$ . Hence,  $j < r < k$ . Apply [13, Lemma 12.12, p. 237] to get that  $U_r \cap B \neq \emptyset$ . The condition  $\delta < \varepsilon$  guarantees that  $U_r \subseteq B(\varepsilon, a)$ . Therefore,  $B(\varepsilon, a) \cap B \neq \emptyset$ . Thus,  $a$  is an element of closure of  $B$ , and so  $A \subseteq B$ .

In conclusion, if  $B \in \mu^{-1}(\mu(A))$ , then either  $B \subseteq A \cup C$  or  $B \subseteq A \cup D$ . This proves that  $X - A = C \parallel D$ . Therefore, in light of Theorem 3.4,  $A$  makes a hole in  $C(X)$ .  $\square$

Now, the next example shows that the assumption chainable in Theorem 5.1 can not be weakened by exhibiting an irreducible continuum  $X$  having an arc  $A$  such that  $X - A$  is not connected and  $A$  does not make a hole in  $C(X)$ .

**Example 5.2.** In the Euclidean plane, let  $X$  be the continuum consisting of a triod  $T$  given by the arc  $[-1, 0] \times \{0\}$  and the arc  $\{0\} \times [-1, 1]$  together with a half-line  $\{(x, \sin(x^{-1})) : 0 < x \leq 1\}$ . Let  $A$  be an arc given by  $\{0\} \times [-2^{-1}, 2^{-1}]$ . See Fig. 2.

Observe that  $X$  is an irreducible continuum and using Proposition 2.1, we have that  $X$  has property (b). Now, if  $U$  and  $V$  are disjoint open subsets such that  $X - A = U \cup V$ , then either  $U = [-1, 0) \times \{0\}$  and  $V = X - (A \cup ([-1, 0) \times \{0\}))$  or  $V = [-1, 0) \times \{0\}$  and  $U = X - (A \cup ([-1, 0) \times \{0\}))$  (see [13, Theorem 11.7, p. 199]). In order to prove that there exists  $B \in \mu^{-1}(\mu(A))$  such that  $B \cap U \neq \emptyset$  and  $B \cap V \neq \emptyset$ , set  $L = \{0\} \times [0, 2^{-1}]$ . Define  $g : [0, 1] \rightarrow C(X)$  by  $g(t) = ([-t, 0] \times \{0\}) \cup (\{0\} \times [-t, (1+t)2^{-1}])$  to get a map satisfying that  $g(0) = L$ ,  $g(1) = T$ ,  $g(0) \subsetneq A \subsetneq g(1)$  and  $g(0) \subsetneq g(t) \subsetneq g(1)$  for each  $t \in (0, 1)$ . By the continuity of  $\mu \circ g$ , there exists  $s \in (0, 1)$  such that  $g(s) \in \mu^{-1}(\mu(A))$ . Thus,  $g(s) \in C(X)$  is such that  $\mu(g(s)) = \mu(A)$ ,  $g(s) \cap U \neq \emptyset$  and  $g(s) \cap V \neq \emptyset$ . Apply Theorem 3.4 to conclude that  $A$  does not make in hole a  $C(X)$ .

The proof of the result below is a consequence of the Theorem 3.4 and [13, Theorem 11.6, p. 199].

**Corollary 5.3.** *Let  $X$  be an irreducible continuum having property (b). If  $A$  makes a hole in  $C(X)$ , then  $A \cap \text{irr}(X) = \emptyset$ .*

## Acknowledgement

The second author thanks CONACYT for financial support (grant number 707713). The authors thank the referee for several suggestions and comments which we feel have made the paper more readable.

## References

- [1] J.G. Anaya, Making holes in hyperspaces, *Topol. Appl.* 154 (2007) 2000–2008.
- [2] J.G. Anaya, Making holes in hyperspaces of subcontinua of a Peano continuum, *Topol. Proc.* 37 (2011) 1–14.
- [3] J.G. Anaya, E. Castañeda-Alvarado, A. Fuentes-Montes de Oca, Making holes in the hyperspace suspension, *Topol. Appl.* 265 (2019) 106816.
- [4] J.G. Anaya, E. Castañeda-Alvarado, F. Orozco-Zitli, Making holes in hyperspaces of subcontinua of some continua, *Adv. Pure Math.* 2 (2012) 133–138.

- [5] J.G. Anaya, A. Fuentes-Montes de Oca, F. Orozco-Zitli, Making holes in the cone, suspension and hyperspaces of some continua, *Comment. Math. Univ. Carol.* 59 (3) (2018) 343–364.
- [6] J.G. Anaya, D. Maya, F. Orozco-Zitli, Making holes in the second symmetric product of dendrites and some fans, *Cienc. Ergo Sum* 19 (1) (2012) 83–92.
- [7] J.G. Anaya, D. Maya, F. Orozco-Zitli, Making holes in the second symmetric product of unicoherent locally connected continua, *Topol. Proc.* 48 (2016) 251–259.
- [8] J. Camargo, D. Maya, L. Ortiz, The hyperspace of nonblockers of  $F_1(X)$ , *Topol. Appl.* 251 (2019) 70–81.
- [9] F. Capulín, L. Juárez-Villa, F. Orozco-Zitli, General properties of pseudo-contractibility, *Topol. Appl.* 247 (2018) 57–71.
- [10] R. Escobedo, C. Estrada-Obregón, J. Sánchez-Martínez, On hyperspace of non-cut sets of continua, *Topol. Appl.* 217 (2017) 97–106.
- [11] C.B. Hughes, Some properties of Whitney continua in the hyperspace  $C(X)$ , *Topol. Proc.* 1 (1976) 209–219.
- [12] A. Illanes, S.B. Nadler Jr., *Hyperspaces: Fundamentals and Recent Advances*, Marcel Dekker, Inc., New York, 1999.
- [13] S.B. Nadler Jr., *Continuum Theory. An Introduction*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 158, Marcel Dekker, Inc., New York, 1992.
- [14] G.T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ., New York, 1942.