Topology and its Applications 292 (2021) 107635

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# Dynamics with Jones' set function $\mathcal{T}$

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ARTICLE INFO

*Article history:*
Received 25 July 2020
Received in revised form 8 February 2021
Accepted 15 February 2021
Available online 19 February 2021

*MSC:*
37B45
54B20
54C60

*Keywords:*
Compactum
Continuum
Dynamics
Irreducible map
Jones' set function $\mathcal{T}$
Minimal map
Mixing map
Transitive map

ABSTRACT

Given a compact metric space $X$ and a continuous function $f\colon X \to X$, we introduce the set function $\mathcal{T}_f\colon X \to 2^X$ given by $\mathcal{T}_f(x) = \mathcal{T}(\{f(x)\})$, where $2^X$ is the family of nonempty closed subsets of $X$ and $\mathcal{T}$ is Jones' set function. We consider the dynamical system $(X, \mathcal{T}_f)$ and study its properties.

## 1. Introduction

Given a compact metric space $X$ and a continuous function $f\colon X \to X$, we introduce the set function $\mathcal{T}_f\colon X \to 2^X$ given by $\mathcal{T}_f(x) = \mathcal{T}(\{f(x)\})$, where $2^X$ is the hyperspace of nonempty closed subsets of $X$ and $\mathcal{T}$ is Jones' set function. We consider the dynamical system $(X, \mathcal{T}_f)$ and study its properties. As far as we know, dynamical systems with set valued functions have been to study relationships between the dynamical properties of a continuum and those of the generalized inverse limit of such continuum with a

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https://doi.org/10.1016/j.topol.2021.107635

set valued function (see [2] and [10], where they work with the unit interval). It seems not very interesting to try to study dynamic properties using Jones' set function $\mathcal{T}$ on a continuum $X$ since it has the property that $A \subset \mathcal{T}(A)$ for every subset $A$ of $X$ [15, Remark 3.1.5]. However if we "compose" the set function $\mathcal{T}$ with a map $f: X \to X$ to form the set valued function $\mathcal{T}_f$, this property is, in general, no longer true and some interesting things happen as we describe below. Note that we obtain different set valued functions with different maps $f$.

The paper is divided in ten sections. After this Indroduction 1 and the section of Definitions 2, we have the Section 3, here we present topological properties of the set functions $\mathcal{T}_f$. In Section 4, we introduce the dynamical properties we investigate for both, the map $f$ and the set function $\mathcal{T}_f$. In Section 5, we give relationships between the dynamical properties of $f$ and $\mathcal{T}_f$. In particular, we show that if $f$ has certain dynamical property, then $\mathcal{T}_f$ also has that property. We give an example (Proposition 5.4) that shows that result of Banks et al. [3, Theorem] which says that transitivity and set of periodic points dense imply sensitivity is not true for the set function $\mathcal{T}_f$. We also extend some results known for a map $f$ to the set function $\mathcal{T}_f$. In Section 6, we characterize indecomposable continua in terms of dynamical properties of the set functions $\mathcal{T}_f$. In Section 7, we consider continua $X$ for which the family $\mathcal{G} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ forms a continuous decomposition of $X$. Given a map $f: X \to X$ with the property that for each $x \in X$, there exists $z \in X$ such that $f(\mathcal{T}(\{x\})) \subset \mathcal{T}(\{z\})$, we consider the induced map to the quotient space of $f$, $F: X/\mathcal{G} \to X/\mathcal{G}$, we prove that $F$ is (dense or closed) minimal if and only if $\mathcal{T}_f$ is (dense or closed, respectively) minimal (Theorems 7.3 and 7.8, respectively). In Section 8, we consider when the set function $\mathcal{T}_{1_X}$ is dense minimal and note that when the continuum is $n$-indecomposable, then $\mathcal{T}_{1_X}$ is dense mimimal (Proposition 8.2). In Section 9, we give an example of a map $\varphi$ such that $\varphi$ is sensitive and the set function $\mathcal{T}_\varphi$ is not (Proposition 9.3). In Section 10, we consider irreducible continua $X$ and the set function $\mathcal{T}_{1_X}$, where $1_X: X \to X$ is the identity map. We show that $\mathcal{T}_{1_X}$ is exact if and only if there exists a positive integer $n$ such that $X$ is $n$-indecomposable (Theorem 10.1). We prove that if $X$ is an irreducible continuum such that $\mathcal{T}_{1_X}$ is transitive, then $X$ must contain an indecomposable continuum with nonempty interior (Theorem 10.4). We give a sufficient condition in order to have that an irreducible continuum is an $\omega$-indecomposable continuum (Theorem 10.8).

## 2. Definitions

If $Z$ is a metric space, then given a subset $A$ of $Z$, the interior of $A$ is denoted by $Int_Z(A)$ and the boundary of $A$ is denoted by $Bd_Z(A)$. If $z$ is a point of $Z$ and $\varepsilon > 0$, then $\mathcal{V}_\varepsilon(z) = \{z' \in Z \mid d(z, z') < \varepsilon\}$.

A *map* is a continuous function. A surjective map $f: Z \to Y$ between metric spaces is *monotone* provided that $f^{-1}(C)$ is connected for every connected subset $C$ of $Y$. If $Z$ is a metric space, $1_Z$ denotes the identity map on $Z$.

A *compactum* is a compact metric space. A *continuum* is a connected compactum. A continuum is *decomposable* if it is the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A continuum $X$ is *aposyndetic* provided that for each pair of points $x_1$ and $x_2$ of $X$ with $x_1 \neq x_2$, there exists a subcontinuum $W$ of $X$ such that $x_1 \in Int_X(W) \subset W \subset X \setminus \{x_2\}$.

Given a continuum $X$, a decomposition $\mathcal{G}$ of $X$ and $q: X \to X/\mathcal{G}$ the quotient map, we say that $\mathcal{G}$ is a *continuous decomposition* provided that $q$ is both open and closed.

The symbols $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively.

## 3. The function $\mathcal{T}_f$

Given a compactum $X$, $\mathcal{P}(X)$ and $2^X$ denote the power set of $X$ and the collection of all closed and nonempty subsets of $X$, respectively. $2^X$ is topologized by the Hausdorff metric. The set $\mathcal{C}(X)$ is the family of all subcontinua of $X$, and it is called the *hyperspace of subcontinua* of $X$.

Let $\mathcal{T}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ be the set function defined by

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there exists } W \in \mathcal{C}(X) \text{ such that } x \in Int_X(W) \subseteq W \subseteq X \setminus A\},$$

for each $A \in \mathcal{P}(X)$. This function is called the *Jones' set function* $\mathcal{T}$. If $A \subseteq X$, $f(A) = \{f(x) \mid x \in A\}$. It is well known that, when $X$ is a continuum, $\mathcal{T}(\mathcal{C}(X)) \subseteq \mathcal{C}(X)$ [15, Theorem 3.1.21].

Given a compactum $X$ and a map $f\colon X \to X$, we define $\mathcal{T}_f\colon X \to \mathcal{C}(X)$ by

$$\mathcal{T}_f(x) = \mathcal{T}(\{f(x)\}), \text{ for each } x \in X.$$

We say that $\mathcal{T}_f$ is onto if $\bigcup_{x\in X} \mathcal{T}_f(x) = X$. Observe that if $f$ is an onto map, then $\mathcal{T}_f$ is onto. However, note that if $\mathcal{C}$ is the Cantor set, $X$ is the cone over $\mathcal{C}$, with vertex $\nu$, and $f\colon X \to X$ is given by $f(x) = \nu$, for each $x \in X$, then $f$ is not onto but $\mathcal{T}_f$ is onto [15, Example 3.1.15].

Let $W$ be a nonempty subset of $X$. We define

$$\mathcal{T}_f(W) = \bigcup\{\mathcal{T}_f(w) \mid w \in W\}.$$

Since $\mathcal{T}$ is upper semicontinuous [15, Theorem 3.3.1] and $f$ is a map, we have that $\mathcal{T}_f$ is upper semicontinuous. Hence, $\mathcal{T}_f(A)$ is a closed subset of $X$ whenever $A$ is closed. Also, if $A$ is a subcontinuum of $X$, since $f(A)$ is a continuum and for each $a \in A$, $\mathcal{T}_f(a) \cap f(A) \neq \emptyset$, we have that $\mathcal{T}_f(A) \in \mathcal{C}(X)$.

Inductively, we define $\mathcal{T}_f^{n+1}\colon X \to \mathcal{C}(X)$ by

$$\mathcal{T}_f^{n+1}(x) = \mathcal{T}_f(\mathcal{T}_f^{n}(x)), \text{ for each } x \in X \text{ and every } n \in \mathbb{N}.$$

Also, if $W$ is a nonempty subset of $X$, then

$$\mathcal{T}_f^{n+1}(W) = \mathcal{T}_f(\mathcal{T}_f^{n}(W)) \text{ for every } n \in \mathbb{N}.$$

Define $\mathcal{T}_f^0\colon X \to \mathcal{C}(X)$ by $\mathcal{T}_f^0(x) = \{x\}$ and $\mathcal{T}_f^0(W) = W$.

**3.1 Lemma.** *Let $X$ be a compactum, let $f\colon X \to X$ be a map and let $W$ be a nonempty subset of $X$. If $w \in W$, then $\mathcal{T}_f^n(w) \subset \mathcal{T}_f^n(W)$, for all $n \in \mathbb{N} \cup \{0\}$.*

**Proof.** We do the proof by induction. For $n = 0$, the lemma is clear. Let $w \in W$ and suppose that $\mathcal{T}_f^n(w) \subset \mathcal{T}_f^n(W)$. By definition and our hypothesis, we have $\mathcal{T}_f^{n+1}(w) = \mathcal{T}_f(\mathcal{T}_f^n(w)) \subset \mathcal{T}_f(\mathcal{T}_f^n(W)) = \mathcal{T}_f^{n+1}(W)$. □

**3.2 Lemma.** *Let $X$ be a compactum, let $f\colon X \to X$ be a map and let $W$ be a nonempty subset of $X$. If $x \in \mathcal{T}_f^n(W)$, then there exists $w \in W$ such that $x \in \mathcal{T}_f^n(w)$, for each $n \in \mathbb{N} \cup \{0\}$.*

**Proof.** We prove the lemma by induction. The result is clear for $n = 0$. Suppose that if $x' \in \mathcal{T}_f^n(W)$, then there exists $w' \in W$ such that $x' \in \mathcal{T}_f^n(w')$. Let $x \in \mathcal{T}_f^{n+1}(W)$. Then $x \in \mathcal{T}_f(\mathcal{T}_f^n(W))$. Thus, there exists $x_1 \in \mathcal{T}_f^n(W)$ such that $x \in \mathcal{T}_f(x_1)$. Since $x_1 \in \mathcal{T}_f^n(W)$, by our hypothesis, there exists $w \in W$ such that $x_1 \in \mathcal{T}_f^n(w)$. This implies, by Lemma 3.1, that $\mathcal{T}_f(x_1) \subset \mathcal{T}_f(\mathcal{T}_f^n(w)) = \mathcal{T}_f^{n+1}(w)$. Therefore, $x \in \mathcal{T}_f^{n+1}(w)$. □

As a consequence of Lemmas 3.1 and 3.2, we obtain:

**3.3 Corollary.** *Let $X$ be a compactum, let $f\colon X \to X$ be a map and let $W$ be a nonempty subset of $X$. Then $\mathcal{T}_f^n(W) = \bigcup\{\mathcal{T}_f^n(w) \mid w \in W\}$.*

**3.4 Corollary.** *Let $X$ be a compactum, let $f\colon X \to X$ be a map, let $n \in \mathbb{N}$ and let $A$ and $B$ be nonempty subsets of $X$. Then $\mathcal{T}_f^n(A \cup B) = \mathcal{T}_f^n(A) \cup \mathcal{T}_f^n(B)$.*

**Proof.** By Corollary 3.3, $\mathcal{T}_f^n(A \cup B) = \bigcup\{\mathcal{T}_f^n(x) \mid x \in A \cup B\} = (\bigcup\{\mathcal{T}_f^n(a) \mid a \in A\}) \cup (\bigcup\{\mathcal{T}_f^n(b) \mid b \in B\}) = \mathcal{T}_f^n(A) \cup \mathcal{T}_f^n(B)$. □

**3.5 Remark.** As a consequence of [15, Theorem 3.1.28] and Corollary 3.3, we have that if $X$ is an aposyndetic continuum, then $\mathcal{T}_f$ and $f$ are essentially the same. Hence, the dynamic behavior of $\mathcal{T}_f$ and $f$ are also the same.

**3.6 Proposition.** *Let $X$ be a continuum and let $f\colon X \to X$ be a map. If $W$ is a nonempty subset of $X$ and $n \in \mathbb{N}$, then $f^n(W) \subset \mathcal{T}_f^n(W)$.*

**Proof.** Let $W$ be a nonempty subset of $X$ and let $w \in W$. We do the proof by induction. For $n = 1$, we have that $f(w) \in \mathcal{T}(\{f(w)\}) = \mathcal{T}_f(w)$, [15, Remark 3.1.5]. Now, suppose that $f^{n-1}(w) \in \mathcal{T}_f^{n-1}(w)$. Hence, $\mathcal{T}_f(f^{n-1}(w)) \subset \mathcal{T}_f(\mathcal{T}_f^{n-1}(w)) = \mathcal{T}_f^n(w)$. Since $\mathcal{T}_f(f^{n-1}(w)) = \mathcal{T}(\{f^n(w)\})$ and $f^n(w) \in \mathcal{T}(\{f^n(w)\})$, we have that $f^n(w) \in \mathcal{T}_f^n(w)$. By Corollary 3.3, $\mathcal{T}_f^n(w) \subset \mathcal{T}_f^n(W)$. Therefore, $f^n(W) \subset \mathcal{T}_f^n(W)$. □

Let $a\mathcal{T}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ be the set function defined by

$$a\mathcal{T}(A) = X \setminus \Bigg\{ x \in X \mid \text{there exists a finite family, } \{W_j\}_{j=1}^n, \text{ of subcontinua of } X \text{ such that } x \in Int_X\left(\bigcap_{j=1}^n W_j\right) \text{ and } \left(\bigcap_{j=1}^n W_j\right) \cap A = \emptyset \Bigg\},$$

for each $A \in \mathcal{P}(X)$. This function is called the *set function* $a\mathcal{T}$.

**3.7 Remark.** Let $X$ be a continuum. Note that, by [4, Lemma 5], if $A$ is a closed subset of $X$, then $\mathcal{T}_{1_X}(A) = a\mathcal{T}(A)$. Also, by [4, Lemma 6], if $X$ is $\mathcal{T}$-additive, then $\mathcal{T}(A) = a\mathcal{T}(A)$ for all closed subsets $A$ of $X$. Hence, by [15, Theorem 3.1.48], if $X$ is a $\mathcal{T}$-additive continuum and $\{A_\lambda\}_{\lambda\in\Lambda}$ is a family of closed subsets of $X$ such that $\bigcup\{A_\lambda \mid \lambda \in \Lambda\}$ is closed in $X$, then $\mathcal{T}_{1_X}(\bigcup\{A_\lambda \mid \lambda \in \Lambda\}) = \bigcup\{\mathcal{T}_{1_X}(A_\lambda) \mid \lambda \in \Lambda\}$.

## 4. Definitions of dynamical properties

Let $X$ be a compactum and let $f\colon X \to X$ be a map. For each $x \in X$, we define the *orbit* of $x$ under $f$ as the set

$$\mathcal{O}(x, f) = \{f^n(x) \mid n \in \mathbb{N} \cup \{0\}\}.$$

We say that:

- A point $x$ of $X$ has a *dense orbit under* $f$ if for each nonempty open subset $U$ of $X$, there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$.
- A point $x$ of $X$ is a *periodic point of* $f$, provided that there exists $m \in \mathbb{N}$ such that $x = f^m(x)$. The periodic point $x$ is of *period* $n$ if $n = \min\{m \in \mathbb{N} \mid x = f^m(x)\}$.

- $f$ is *backward dense minimal* if the for each point $x$ of $X$, the set $\mathcal{J}(x) = \{x' \in X \mid f^n(x') = x$ for some $n \in \mathbb{N}\}$ is dense in $X$.

- $f$ is *exact* provided that for each nonempty open subset $U$ of $X$, there exists $n \in \mathbb{N}$ such that $f^n(U) = X$.

- $f$ has *sensitive dependence on initial conditions* if there exists $\varepsilon > 0$ such that for every point $x_1$ of $X$ and each $\delta > 0$, there exist $x_2 \in \mathcal{V}_\delta(x_1)$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \varepsilon$. We write $f$ is *sensitive*, for short.

- $f$ is *irreducible*, provided that if $A$ is a closed subset of $X$ such that $f(A) = X$, then $A = X$.

- $f$ is *closed minimal*, if $A$ is a nonempty closed subset of $X$ such that $f(A) \subset A$, then $A = X$.

- $f$ is *dense minimal*, provided that each point of $X$ has a dense orbit under $f$.

- $f$ is *mixing*, if for every pair of nonempty open subsets $U$ and $V$ of $X$, there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

- $f$ is *strongly transitive*, provided that for each nonempty open subset $U$ of $X$, there exists $n \in \mathbb{N}$ such that $X = \bigcup_{k=0}^{n} f^k(U)$.

- $f$ is *totally transitive*, if $f^n$ is transitive for all $n \in \mathbb{N}$.

- $f$ is *transitive*, provided that for each pair of nonempty open subsets $U$ and $V$ of $X$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.

**4.1 Remark.** It is well known that a map defined between compacta, the map is dense minimal if and only if it is closed minimal [13, pp. 85-86]. Thus, we say that $f$ is minimal instead of either closed minimal or dense minimal. In the case of $\mathcal{T}_f$ maps, the concepts of closed minimal and dense minimal are not equivalent, as we show below. Also, for maps the concept of backward dense minimal is just called backward minimal, we add the word dense in order to be related to what we call dense minimal. In the following diagrams, we represent all the interrelations between these classes of maps.

The following diagram comprises all possible relationships between the classes of maps defined above. An arrow means inclusion.

$$\begin{tikzcd}
f \text{ exact} \arrow[r] \arrow[d] & f \text{ mixing} \arrow[d] & \\
f \text{ strongly transitive} \arrow[d] & f \text{ totally transitive} \arrow[d] & \\
f \text{ backward dense minimal} \arrow[r] & f \text{ transitive} & f \text{ minimal} \arrow[l]
\end{tikzcd}$$

Diagram 1

The goal of this paper is to study dynamical properties of the function $\mathcal{T}_f$. We introduce the following definitions:

For each $x \in X$, we define the *orbit* of $x$ under $\mathcal{T}_f$ as the set

$$\mathcal{O}(x, \mathcal{T}_f) = \{x' \in X \mid \text{there exists } n \in \mathbb{N} \cup \{0\} \text{ such that } x' \in \mathcal{T}_f^n(x)\}.$$

We say that:

- A point $x$ of $X$ has a *dense orbit under* $\mathcal{T}_f$ if for each nonempty open subset $U$ of $X$, there exists $n \in \mathbb{N}$ such that $\mathcal{T}_f^n(x) \cap U \neq \emptyset$.

- A point $x$ of $X$ is *periodic point of* $\mathcal{T}_f$, provided that there exists $m \in \mathbb{N}$ such that $x \in \mathcal{T}_f^m(x)$. The periodic point $x$ is of *period* $n$ if $n = \min\{m \in \mathbb{N} \mid x \in \mathcal{T}_f^m(x)\}$.

- $\mathcal{T}_f$ is *backward dense minimal* if the for each point $x$ of $X$, the set $\mathcal{J}_{\mathcal{T}_f}(x) = \{x' \in X \mid x \in \mathcal{T}_f^n(x'),\ \text{for some } n \in \mathbb{N} \cup \{0\}\}$ is dense in $X$.

- $\mathcal{T}_f$ is *exact*, if for every nonempty open subset $U$ of $X$, there exists $n \in \mathbb{N}$ such that $\mathcal{T}_f^n(U) = X$.

- $\mathcal{T}_f$ has *sensitive dependence on initial conditions*, provided that there exists $\varepsilon > 0$ such that for every point $x_1$ of $X$ and each $\delta > 0$, there exist $x_2 \in \mathcal{V}_\delta(x_1)$ and $n \in \mathbb{N}$ such that $\mathcal{H}(\mathcal{T}_f^n(x_1), \mathcal{T}_f^n(x_2)) \geq \varepsilon$, where $\mathcal{H}$ is the Hausdorff metric on $\mathcal{C}(X)$. We write $\mathcal{T}_f$ is *sensitive* for short.

- $\mathcal{T}_f$ is *irreducible*, provided that if $A$ is a closed subset of $X$ such that $\mathcal{T}_f(A) = X$, then $A = X$.

- $\mathcal{T}_f$ is *closed minimal*, if $A$ is a nonempty closed subset of $X$ such that $\mathcal{T}_f(A) \subset A$, then $A = X$.

- $\mathcal{T}_f$ is *dense minimal*, provided that each point of $X$ has a dense orbit under $\mathcal{T}_f$.

- $\mathcal{T}_f$ is *mixing*, if for every pair of nonempty open subsets $U$ and $V$ of $X$, there exists $N \in \mathbb{N}$ such that $\mathcal{T}_f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

- $\mathcal{T}_f$ is *strongly transitive*, provided that for each nonempty open subset $U$ of $X$, there exists $n \in \mathbb{N}$ such that $X = \bigcup_{k=0}^{n} \mathcal{T}_f^k(U)$.

- $\mathcal{T}_f$ is *totally transitive*, if $\mathcal{T}_f^n$ is transitive for all $n \in \mathbb{N}$.

- $\mathcal{T}_f$ is *transitive*, provided that for each pair of nonempty open subsets $U$ and $V$ of $X$, there exists $n \in \mathbb{N}$ such that $\mathcal{T}_f^n(U) \cap V \neq \emptyset$.

The following diagram follows from the definitions, Theorems 5.6, 5.7 and 5.10.

$$\begin{tikzcd}
\mathcal{T}_f \text{ exact} \arrow[r] \arrow[d] & \mathcal{T}_f \text{ mixing} \arrow[d] & \\
\mathcal{T}_f \text{ strongly transitive} \arrow[d] & \mathcal{T}_f \text{ totally transitive} \arrow[d] & \mathcal{T}_f \text{ closed minimal} \\
\mathcal{T}_f \text{ backward dense minimal} \arrow[r] & \mathcal{T}_f \text{ transitive} & \mathcal{T}_f \text{ dense minimal} \arrow[l] \arrow[u]
\end{tikzcd}$$

Diagram 2

**4.2 Question.** Under what conditions on $X$ and/or $f$ we have that the converse implications in Diagram 2 are true?

Partial answers to Question 4.2 are given in Remarks 4.3, and 4.4.

**4.3 Remark.** Given a continuum $X$, by the definitions, we have the following are equivalent:

(1) $\mathcal{T}_{1_X}$ is mixing;
(2) $\mathcal{T}_{1_X}$ is totally transitive;
(3) $\mathcal{T}_{1_X}$ is transitive.

Also note:

**4.4 Remark.** Given a continuum $X$, by the definitions, we have the following are equivalent:

(1) $\mathcal{T}_{1_X}$ is exact;
(2) $\mathcal{T}_{1_X}$ is strongly transitive.

**4.5 Lemma.** *Let $X$ and $Y$ be continua, where $Y$ is aposyndetic, and let $f\colon X \to Y$ be a monotone map. If $A$ is a subset of $X$, then $\mathcal{T}_{1_X}(A) \subset f^{-1}(f(A))$.*

**Proof.** By [14, Lemma 3.2], $\mathcal{T}(\{a\}) \subset f^{-1}(f(a))$ for each $a \in A$. Thus, $\mathcal{T}_{1_X}(A) = \bigcup\{\mathcal{T}(\{a\}) \mid a \in A\} \subset \bigcup\{f^{-1}(f(a)) \mid a \in A\} = f^{-1}(f(A))$. □

We finish this section with an interesting observation related with monotone maps and aposyndetic continua.

**4.6 Theorem.** *Let $X$ and $Y$ be nondegenerate continua and let $f\colon X \to Y$ be a monotone map. If $\mathcal{T}_{1_X}$ is transitive, then $Y$ is not aposyndetic.*

**Proof.** Suppose that $Y$ is aposyndetic. Let $U$ and $V$ be disjoint nonempty open subsets of $Y$. By Lemma 4.5, $\mathcal{T}_{1_X}(f^{-1}(U)) \subset f^{-1}(f(f^{-1}(U))) = f^{-1}(U)$. Hence, $\mathcal{T}_{1_X}^k(f^{-1}(U)) \subset f^{-1}(U)$ for each $k \in \mathbb{N}$. Since $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, $\mathcal{T}_{1_X}^n(f^{-1}(U)) \cap f^{-1}(V) = \emptyset$ for each $n \in \mathbb{N}$. Therefore, $\mathcal{T}_{1_X}$ is not transitive. □

## 5. General properties

In this section, we present relationships between the maps $f$ and $\mathcal{T}_f$; i.e., it is natural to ask that if $f$ satisfies some property, then is it true that $\mathcal{T}_f$ satisfies the same property, and vice versa?

A continuum $X$ has the *property of Kelley*, provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_1$ and $x_2$ are points of $X$, $d(x_1, x_2) < \delta$ and $K$ is a subcontinuum of $X$ containing $x_1$, then there exists a subcontinuum $L$ of $X$ containing $x_2$ such that $\mathcal{H}(K, L) < \varepsilon$.

**5.1 Theorem.** *Let $X$ be a decomposable continuum. If $\mathcal{T}_{1_X}$ is transitive, then $X$ does not have the property of Kelley.*

**Proof.** Assume $X$ has the property of Kelley. Let $A$ be a proper subcontinuum with nonempty interior [15, Theorem 1.7.25], and let $U$ be a nonempty open subset of $X$ such that $Cl_X(U) \cap A = \emptyset$. Since $\mathcal{T}_{1_X}$ is transitive, there exists $k \in \mathbb{N}$ such that $\mathcal{T}_{1_X}^k(U) \cap Int_X(A) \neq \emptyset$. Hence, $\mathcal{T}_{1_X}^k(Cl_X(U)) \cap A \neq \emptyset$.

Since $X$ has the property of Kelley, $A$ is a subcontinuum of $X$ with nonempty interior and $Cl_X(U) \cap A = \emptyset$, by [16, Corollary 4.2], there exists a subcontinuum $K_1$ of $X$ such that $A \subset Int_X(K_1)$ and $K_1 \cap Cl_X(U) = \emptyset$. Hence, $\mathcal{T}(Cl_X(U)) \cap A = \emptyset$. Since $\mathcal{T}_{1_X}(Cl_X(U)) \subset \mathcal{T}(Cl_X(U))$, we have that $\mathcal{T}_{1_X}(Cl_X(U)) \cap A = \emptyset$. By a similar argument, there exists a subcontinuum $K_2$ of $X$ such that $K_1 \subset Int_X(K_2)$ and $K_2 \cap \mathcal{T}(Cl_X(U)) = \emptyset$. Thus, $\mathcal{T}_{1_X}^2(Cl_X(U)) \cap A = \emptyset$. Continuing with this process, we obtain that for every $n \geq 3$, there exists a subcontinuum $K_n$ of $X$ such that $K_{n-1} \subset Int_X(K_n)$ and $K_n \cap \mathcal{T}^{n-1}(Cl_X(U)) = \emptyset$. Thus, $\mathcal{T}_{1_X}^n(Cl_X(U)) \cap A = \emptyset$, a contradiction to the transitivity of $\mathcal{T}_{1_X}$. Therefore, $X$ does not have the property of Kelley. □

The following theorem is obtained from Proposition 3.6.

**5.2 Theorem.** *Let $X$ be a compactum, let $x \in X$ and let $f\colon X \to X$ be a map.*

(1) *If $x$ has a dense orbit under $f$, then $x$ has a dense orbit under $\mathcal{T}_f$.*
(2) *If $x$ is a periodic point of $f$, then $x$ is a periodic point of $\mathcal{T}_f$.*
(3) *If the set of periodic points of $f$ is dense in $X$, then the set of periodic points of $\mathcal{T}_f$ is dense in $X$.*

The next theorem follows from the definitions.

**5.3 Theorem.** *Let $\mathcal{A}$ be a class of maps. Let $X$ be a continuum and let $f\colon X \to X$ be a map. Assume $\mathcal{A}$ is any of the following classes of maps: backward dense minimal, exact, mixing, transitive, strongly transitive or totally transitive. If $f$ belongs to $\mathcal{A}$, then $\mathcal{T}_f$ also belongs to $\mathcal{A}$.*

Let $\mathcal{C} \subset [0,1]$ be the Cantor set. Let $\nu = (\frac{1}{2}, 1)$ and let $F_{\mathcal{C}} = \{\nu t + x(1-t) \mid t \in [0,1], x \in \mathcal{C} \times \{0\}\}$; $F_{\mathcal{C}}$ is the cone over the Cantor set, and it is also known as the *Cantor fan.*

**5.4 Proposition.** *There exist onto maps $H, G, L\colon F_{\mathcal{C}} \to F_{\mathcal{C}}$ such that:*

1. *$H$ and $G$ are not transitive;*
2. *$L$ is not sensitive;*
3. *$\mathcal{T}_H$ and $\mathcal{T}_G$ are exact, but not sensitive;*
4. *$\mathcal{T}_L$ is sensitive, but not transitive;*
5. *$\mathcal{T}_H$ and $\mathcal{T}_L$ are not closed minimal;*
6. *$\mathcal{T}_G^2(x) = F_{\mathcal{C}}$ for each $x \in F_{\mathcal{C}}$; hence, $\mathcal{T}_G$ is dense minimal;*
7. *$x \in \mathcal{T}_H(x)$ for each $x \in F_{\mathcal{C}}$; i.e., every point is fixed of $\mathcal{T}_H$;*
8. *$x \in \mathcal{T}_G^2(x)$ for each $x \in F_{\mathcal{C}}$; i.e., every point is of period at most 2;*
9. *$\nu$ is the only periodic point of $\mathcal{T}_L$ and $\nu \in \mathcal{T}_L(\nu)$.*

**Proof.** Let $g, h\colon [0,1] \to [0,1]$ be defined by

$$h(t) = \begin{cases} 2t, & \text{if } t \in [0, \frac{1}{2}]; \\ 1, & \text{if } t \in [\frac{1}{2}, 1], \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 1-4t, & \text{if } t \in [0, \frac{1}{4}]; \\ 4t-1, & \text{if } t \in [\frac{1}{4}, \frac{1}{2}]; \\ 1, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Let $\ell\colon \mathcal{C} \to \mathcal{C}$ be a minimal and expansive homeomorphism (see [1, Example 6.14]).

Let $H, G, L\colon F_{\mathcal{C}} \to F_{\mathcal{C}}$ be given by

$$H(\nu t + x(1-t)) = \nu h(t) + x(1-h(t)),\ G(\nu t + x(1-t)) = \nu g(t) + x(1-g(t)),$$

and

$$L(\nu t + x(1-t)) = \nu t + \ell(x)(1-t),$$

for each $x \in \mathcal{C} \times \{0\}$ and $t \in [0,1]$. Note that the maps $H, G$ and $L$ satisfy the above statements. □

Proposition 5.4 shows that the converse implications in Theorems 5.2 and 5.3 are not necessarily true. By Proposition 5.4 (6), $x$ has a dense orbit under $\mathcal{T}_G$ for every $x \in F_{\mathcal{C}}$, but $G(\nu) = \nu$. Hence, $\nu$ does not have a dense orbit under $G$. The converse implication of Theorem 5.2 (1) is not true. By items (7) and (8) of Proposition 5.4, each point of $F_{\mathcal{C}}$ is periodic under $\mathcal{T}_H$ and $\mathcal{T}_G$, respectively. Observe that

$$H(\{\nu t + x(1-t) \mid t \in (\tfrac{1}{2}, 1), x \in \mathcal{C} \times \{0\}\}) = G(\{\nu t + x(1-t) \mid t \in (\tfrac{1}{2}, 1), x \in \mathcal{C} \times \{0\}\}) = \{\nu\}.$$

Thus, the converse of Theorem 5.2 (2) and (3) are false.

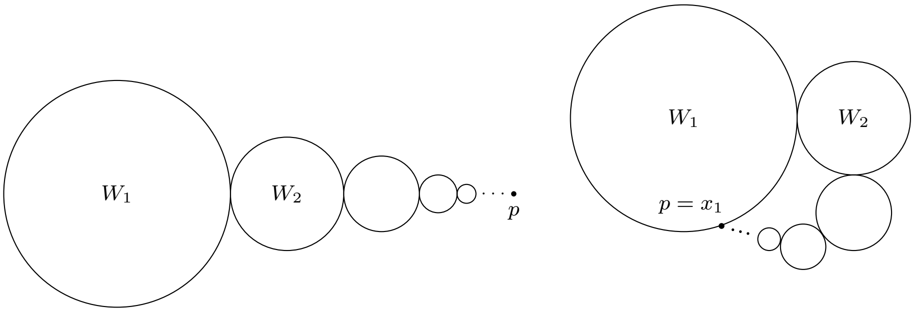

**Fig. 1.** Continua $X$ and $Y$.

By Proposition 5.4 (1), $H$ and $G$ are not transitive. Hence, they are not backward dense minimal, exact, mixing, strongly transitive or totally transitive, by Diagram 1. $\mathcal{T}_H$ and $\mathcal{T}_G$ are exact, by Proposition 5.4 (3). Thus, they are backward dense minimal, exact, mixing, strongly transitive and totally transitive, by Diagram 2. We do not have all the converse implications of Theorem 5.3.

**5.5 Remark.** In [3, Theorem], it is shown that if $f$ is transitive and the set of periodic points of $f$ is dense, then $f$ is sensitive. Proposition 5.4 shows that $\mathcal{T}_H$ and $\mathcal{T}_G$ are transitive, the sets of periodic points of $\mathcal{T}_H$ and of $\mathcal{T}_G$ are dense, but they are not sensitive.

**5.6 Theorem.** *Let $X$ be a compactum and let $f\colon X \to X$ be a map. If $\mathcal{T}_f$ is strongly transitive, then $\mathcal{T}_f$ is backward dense minimal.*

**Proof.** Let $U$ be a nonempty open subset of $X$. Since $\mathcal{T}_f$ is strongly transitive, there exists $n \in \mathbb{N}$ such that $\bigcup_{k=0}^{n} \mathcal{T}_f^k(U) = X$. Let $x \in \mathcal{T}_f^k(U)$, for some $k \in \{0, \dots, n\}$. Then, by Corollary 3.3, there exists $u \in U$ such that $x \in \mathcal{T}_f^k(u)$. Hence, $u \in \mathcal{J}_{\mathcal{T}_f}(x) \cap U$. Therefore, $\mathcal{J}_{\mathcal{T}_f}(x)$ is dense in $X$. Since $x$ is an arbitrary point of $X$, $\mathcal{T}_f$ is backward dense minimal. □

**5.7 Theorem.** *Let $X$ be a compactum and let $f\colon X \to X$ be a map. If $\mathcal{T}_f$ is backward dense minimal, then $\mathcal{T}_f$ is transitive.*

**Proof.** Let $U$ and $V$ be nonempty open subsets of $X$. Let $x$ be a point of $V$. Since $\mathcal{J}_{\mathcal{T}_f}(x)$ is dense in $X$, there exist $x' \in U \cap \mathcal{J}_{\mathcal{T}_f}(x)$ and an $n \in \mathbb{N}$ such that $x \in \mathcal{T}_f^n(x')$. Hence, $\mathcal{T}_f^n(U) \cap V \neq \emptyset$. Therefore, $\mathcal{T}_f$ is transitive. □

The following example shows that the converse implications in Theorems 5.6 and 5.7 are not true.

**5.8 Example.** There exist continua $X$ and $Y$ such that:

(1) $\mathcal{T}_{1_X}$ is transitive, but it is not backward dense minimal;
(2) $\mathcal{T}_{1_Y}$ is backward dense minimal, but it is not strongly transitive.

Let $\{W_i\}_{i\in\mathbb{N}}$ be a sequence of indecomposable subcontinua of $\mathbb{R}^2$ such that $|W_i \cap W_j| = 1$ if and only if $|i - j| = 1$, for each $i$ and $j$ in $\mathbb{N}$, and $\lim_{i\to\infty} W_i = \{p\}$, where $p \notin \bigcup_{i\in\mathbb{N}} W_i$. Let $X = (\bigcup_{i\in\mathbb{N}} W_i) \cup \{p\}$. Let $x_1 \in W_1 \setminus W_2$ and let $Y = X/\{x_1, p\}$ (see Fig. 1). Then we have that $X$ and $Y$ satisfy the above statements.

**5.9 Theorem.** *Let $X$ be a compactum and let $f\colon X \to X$ be a map. If $\mathcal{T}_f$ is dense minimal, then $\mathcal{T}_f$ is transitive.*

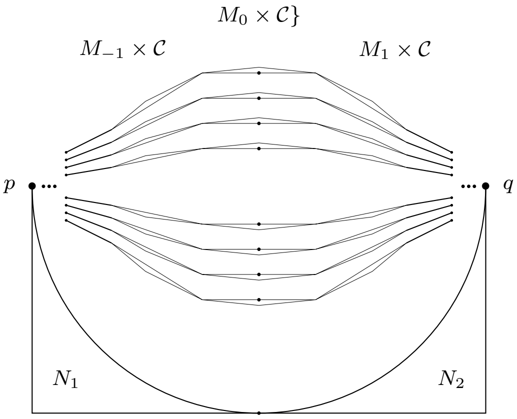

**Fig. 2.** $\mathfrak{T}(X) = \{X\}$, but $\mathcal{T}_{1_X}$ is not closed minimal.

**Proof.** Let $U$ and $V$ be nonempty open subsets of $X$. Let $x \in U$. Since $\mathcal{T}_f$ is dense minimal, there exists $n \in \mathbb{N} \cup \{0\}$ such that $\mathcal{T}_f^n(x) \cap V \neq \emptyset$. By Lemma 3.1, we have that $\mathcal{T}_f^n(U) \cap V \neq \emptyset$. Therefore, $\mathcal{T}_f$ is transitive. □

**5.10 Theorem.** *Let $X$ be a compactum and let $f: X \to X$ be a map. If $\mathcal{T}_f$ is dense minimal, then $\mathcal{T}_f$ is closed minimal.*

**Proof.** Suppose $\mathcal{T}_f$ is not closed minimal. Then there exists a nonempty proper closed subset of $X$ such that $\mathcal{T}_f(A) \subset A$. Let $a \in A$. Then, by Lemma 3.1, for each $n \in \mathbb{N} \cup \{0\}$, $\mathcal{T}_f^n(a) \subset \mathcal{T}_f^n(A) \subset A$. Hence, $X \setminus A$ is a nonempty open subset of $X$ such that $\mathcal{T}_f^n(a) \cap (X \setminus A) = \emptyset$ for all $a \in A$ and every $n \in \mathbb{N}$. Thus, $\mathcal{T}_f$ is not dense minimal. □

Let $X$ be a continuum. A subset $A$ of $X$ is called $\mathcal{T}$*-closed* provided that $\mathcal{T}(A) = A$. The collection of all $\mathcal{T}$-closed subsets of $X$ is denoted by $\mathfrak{T}(X)$.

**5.11 Theorem.** *Let $X$ be a continuum. If $\mathcal{T}_{1_X}$ is closed minimal, then the set $\mathfrak{T}(X) = \{X\}$.*

**Proof.** Suppose that $A \in \mathfrak{T}(X)$ where $A \neq X$. Observe that $\mathcal{T}_{1_X}(a) \subset \mathcal{T}(A) = A$ for each $a \in A$. Hence, $\mathcal{T}_{1_X}(A) \subset A$. Therefore, $\mathcal{T}_{1_X}$ is not closed minimal. □

**5.12 Example.** There exists a continuum $X$ such that $\mathfrak{T}(X) = \{X\}$, but $\mathcal{T}_{1_X}$ is not closed minimal.

Let $\{M_i \mid i \in \mathbb{Z}\}$ be a sequence of indecomposable subcontinua of $\mathbb{R}^2$ such that:

- $|M_i \cap M_j| = 1$ if and only if $|i - j| = 1$;
- $Cl_{\mathbb{R}^2}(\bigcup_{i \in \mathbb{Z}} M_i) = \{p\} \cup (\bigcup_{i \in \mathbb{Z}} M_i) \cup \{q\}$ is irreducible between $p$ and $q$, where $\{p, q\} \cap (\bigcup_{i \in \mathbb{Z}} M_i) = \emptyset$.

Let $Z = \{p\} \cup (\bigcup_{i \in \mathbb{Z}} M_i) \cup \{q\}$. Let $N_1$ and $N_2$ be indecomposable subcontinua of $\mathbb{R}^2$ such that $|N_1 \cap N_2| = 1$, $(N_1 \cap Z) = \{p\}$, $(N_2 \cap Z) = \{q\}$ and $N_1 \cup N_2$ is irreducible between $p$ and $q$. In $(Z \cup N_1 \cup N_2) \times \mathcal{C}$. Define $(x, s) \sim (y, t)$ if and only if $x = y$ and $x \in N_1 \cup N_2$. Let

$$X = ((Z \cup N_1 \cup N_2) \times \mathcal{C}) / \sim \text{ (see Fig. 2).}$$

Observe that $\mathcal{T}_{1_X}(x) \subset N_1 \cup N_2$, whenever $x \in N_1 \cup N_2$. Hence, $\mathcal{T}_{1_X}(N_1 \cup N_2) = N_1 \cup N_2$ and $\mathcal{T}_{1_X}$ is not closed minimal. Note that $\mathcal{T}(N_1 \cup N_2) = X$ and, if $A$ is a nonempty closed subset of $X$ such that

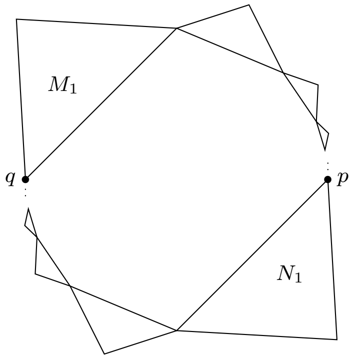

**Fig. 3.** $\mathcal{T}_{1_X}$ is closed minimal.

$A \neq N_1 \cup N_2$, then $\mathcal{T}(A)$ is the union of all indecomposable continua that touch $A$. Hence, $A \subsetneq \mathcal{T}(A)$. Therefore, $\mathfrak{T}(X) = \{X\}$.

**5.13 Question.** Let $X$ be a hereditarily decomposable continuum. If $\mathfrak{T}(X) = \{X\}$, then is $\mathcal{T}_{1_X}$ closed minimal?

**5.14 Question.** Is there a hereditarily decomposable continuum $X$ such that $\mathfrak{T}(X) = \{X\}$?

Given a continuum $X$ and a map $f\colon X \to X$, let $\mathcal{I}_f = \{A \in 2^X \mid f(A) \subset A\}$.

**5.15 Theorem.** *Let $X$ be a continuum and let $f\colon X \to X$ be a map. If $\mathcal{T}_f$ is closed minimal, then $\mathfrak{T}(X) \cap \mathcal{I}_f = \{X\}$.*

**Proof.** Let $A \in \mathfrak{T}(X) \cap \mathcal{I}_f$. Then $f(A) \subset A$. Hence, $\mathcal{T}_f(a) = \mathcal{T}(\{f(a)\}) \subset \mathcal{T}(A) = A$ for each $a \in A$. Thus, $\mathcal{T}_f(A) \subset A$, Corollary 3.3. Since $\mathcal{T}_f$ is closed minimal, $A = X$. □

**5.16 Example.** There exists a continuum $X$ such that $\mathcal{T}_{1_X}$ is closed minimal, but $\mathcal{T}_{1_X}$ is neither dense minimal nor irreducible.

Let $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ be sequences of indecomposable continua, and let $q \in M_1$ and $p \in N_1$ be such that:

- $M_i \cap N_j = \emptyset$ whenever $i \neq j$;
- $|M_i \cap M_j| = 1$ and $|N_i \cap N_j| = 1$ if and only if $|i - j| = 1$;
- $\lim_{n\to\infty} M_n = \{p\}$ and $\lim_{n\to\infty} N_n = \{q\}$;
- $(\bigcup_{n\in\mathbb{N}} M_n) \cup \{p\}$ and $(\bigcup_{n\in\mathbb{N}} N_n) \cup \{q\}$ are irreducible continua.

Let $X = (\bigcup_{n\in\mathbb{N}} M_n) \cup (\bigcup_{n\in\mathbb{N}} N_n)$ (see Fig. 3).

Note that if $x \in \bigcup_{n\in\mathbb{N}} M_n$, then $\mathcal{T}_{1_X}^k(x) \subset \bigcup_{n\in\mathbb{N}} M_n$ for each $k \in \mathbb{N}$. Thus, the orbit of $x$ under $\mathcal{T}_{1_X}$ is not dense. Therefore, $\mathcal{T}_{1_X}$ is not dense minimal.

Let $A$ be a proper closed subset of $X$. Then $\mathcal{T}_{1_X}(A) = (\bigcup\{M_i \mid A \cap M_i \neq \emptyset\}) \cup (\bigcup\{N_i \mid A \cap N_i \neq \emptyset\})$, and $A \subsetneq \mathcal{T}_{1_X}(A)$; i.e., $\mathcal{T}_{1_X}$ is closed minimal. For each $n \in \mathbb{N}$, let $x_n \in M_n$ and $y_n \in N_n$. Then $K = \{x_n \mid n \in \mathbb{N}\} \cup \{y_n \mid n \in \mathbb{N}\} \cup \{p, q\}$ is a proper closed subset of $X$ such that $\mathcal{T}_{1_X}(K) = X$. Thus, $\mathcal{T}_{1_X}$ is not irreducible.

**5.17 Remark.** Note that $\mathcal{T}_{1_{[0,1]}}$ is irreducible, but it is not closed minimal. Also, $\mathcal{T}_{1_X}$ is dense minimal, but it is not irreducible, where $X$ is an indecomposable continuum.

**5.18 Lemma.** *Let $X$ be a continuum and let $f\colon X \to X$ be a map. If $f$ is a homeomorphism, then $\mathcal{T}_f^n(A) = f^n(\mathcal{T}_{1_X}^n(A))$ for each subset $A$ of $X$ and every $n \in \mathbb{N}$.*

**Proof.** Since $f$ is a homeomorphism, $\mathcal{T}_f(A) = \bigcup\{\mathcal{T}(\{f(a)\}) \mid a \in A\} = \bigcup\{f(\mathcal{T}(\{a\}) \mid a \in A)\}$ [15, Lemma 5.1.1]. Also, $\mathcal{T}_f(A) = f(\bigcup\{\mathcal{T}(\{a\}) \mid a \in A\}) = f(\mathcal{T}_{1_X}(A))$. We prove the result is true for $n = 2$, the inductive step is proven similarly. By definition, we have $\mathcal{T}_f^2(A) = \mathcal{T}_f(\mathcal{T}_f(A)) = \mathcal{T}_f(f(\mathcal{T}_{1_X}(A))) = f(\mathcal{T}_{1_X}(f(\mathcal{T}_{1_X}(A)))) = f(\bigcup\{\mathcal{T}(\{z\}) \mid z \in f(\mathcal{T}_{1_X}(A))\}) = f(\bigcup\{\mathcal{T}(\{f(w)\}) \mid w \in \mathcal{T}_{1_X}(A)\}) = f(\bigcup\{f(\mathcal{T}(\{w\})) \mid w \in \mathcal{T}_{1_X}(A)\}) = f(f(\bigcup\{\mathcal{T}(\{w\}) \mid w \in \mathcal{T}_{1_X}(A)\})) = f^2(\mathcal{T}_{1_X}^2(A))$. □

From Lemma 5.18, we obtain:

**5.19 Theorem.** *Let $X$ be a continuum and let $f\colon X \to X$ be a homeomorphism. Then:*

(1) *$\mathcal{T}_{1_X}$ is exact if and only if $\mathcal{T}_f$ is exact.*
(2) *$\mathcal{T}_{1_X}$ is irreducible if and only if $\mathcal{T}_f$ is irreducible.*

## 6. Indecomposable continua

As a consequence of [15, Theorem 3.1.39], we obtain:

**6.1 Theorem.** *If $X$ is an indecomposable continuum and $f\colon X \to X$ is a map, then $\mathcal{T}_f$ is exact.*

**Proof.** Let $U$ be a nonempty open subset of $X$ and let $x \in U$. Note that $\mathcal{T}_f(x) \subset \mathcal{T}_f(U)$. Since $\mathcal{T}_f(x) = \mathcal{T}(\{f(x)\})$ and, by [15, Theorem 3.1.39], $\mathcal{T}(\{f(x)\}) = X$, we have that $\mathcal{T}_f(U) = X$. Therefore, $\mathcal{T}_f$ is exact. □

Given a continuum $X$ and a point $z$ of $X$, we define $c_z\colon X \to X$ as the constant map $c_z(x) = z$ for each $x \in X$. Let

$$\mathcal{E}_X = \{c_{x_0} \mid x_0 \in X\}.$$

**6.2 Theorem.** *Let $X$ be a continuum. Then the following are equivalent:*

(1) *$X$ is indecomposable;*
(2) *$\mathcal{T}_f$ is dense minimal for each map $f$;*
(3) *$\mathcal{T}_f$ is dense minimal for each map $f \in \mathcal{E}_X$;*
(4) *$\mathcal{T}_f$ is closed minimal for each map $f$;*
(5) *$\mathcal{T}_f$ is closed minimal for each map $f \in \mathcal{E}_X$.*

**Proof.** It is clear that 1 implies 2, 1 implies 4, 2 implies 3 and 4 implies 5. 3 implies 5 follows from Theorem 5.10.

We see that 5 implies 1. Let $z \in X$. Note that $\mathcal{T}_{c_z}(A) = \mathcal{T}(\{z\})$ for each $A \in 2^X$. If $\mathcal{T}(\{z\}) \neq X$, then $\mathcal{T}(\{z\})$ is a closed subset of $X$ such that $\mathcal{T}_{c_z}(\mathcal{T}(\{z\})) \subseteq \mathcal{T}(\{z\})$. This contradicts that $\mathcal{T}_{c_z}$ is closed minimal. Thus, $\mathcal{T}(\{z\}) = X$. Since $z$ is an arbitrary point of $X$, $\mathcal{T}(\{x\}) = X$ for each $x \in X$. Therefore, $X$ is indecomposable [15, Theorem 3.1.39]. □

## 7. Examples of minimality of $\mathcal{T}_f$

In this section, we present more examples where the function $\mathcal{T}_f$ is either closed or dense minimal.

**7.1 Notation.** Let $X$ be a continuum such that $\mathcal{G} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ is a continuous decomposition of $X$. Note that $\mathcal{T}(\{x\}) = q^{-1}(q(x))$ for each $x \in X$, where $q\colon X \to X/\mathcal{G}$ is the quotient map. Let $f\colon X \to X$ be a map such that for each $x \in X$, there exists $z \in X$ where $f(\mathcal{T}(\{x\})) \subseteq \mathcal{T}(\{z\})$. Let $F\colon X/\mathcal{G} \to X/\mathcal{G}$ be the map such that

$$F \circ q = q \circ f.$$

Observe that $F$ is well defined continuous [9, Theorem 3.2].

**7.2 Lemma.** *With Notation 7.1. If $\chi \in X/\mathcal{G}$ and $x \in X$ are such that $q(x) = \chi$, then $\mathcal{T}_f^n(x) = q^{-1}(F^n(\chi))$, for each $n \in \mathbb{N}$.*

**Proof.** The proof is done by induction. For $n = 1$, $\mathcal{T}_f(x) = \mathcal{T}(\{f(x)\}) = q^{-1}(q(f(x))) = q^{-1}(F(q(x))) = q^{-1}(F(\chi))$. Assume $\mathcal{T}_f^n(x) = q^{-1}(F^n(\chi))$. Then we have:

$$\begin{aligned}
\mathcal{T}_f^{n+1}(x) &= \mathcal{T}_f(\mathcal{T}_f^n(x)) = \mathcal{T}_f(q^{-1}(F^n(\chi))) \\
&= \bigcup\{\mathcal{T}_f(z) \mid z \in q^{-1}(F^n(\chi))\} \\
&= \bigcup\{\mathcal{T}(\{f(z)\}) \mid z \in q^{-1}(F^n(\chi))\} \\
&= \bigcup\{q^{-1}(q(f(z))) \mid z \in q^{-1}(F^n(\chi))\} \\
&= q^{-1}\left(\bigcup\{q(f(z)) \mid z \in q^{-1}(F^n(\chi))\}\right) \\
&= q^{-1}(\bigcup\{F(q(z)) \mid z \in q^{-1}(F^n(\chi))\}) \\
&= q^{-1}(\{F(F^n(\chi)\}) = q^{-1}(F^{n+1}(\chi)). \quad \square
\end{aligned}$$

**7.3 Theorem.** *With Notation 7.1. $F\colon X/\mathcal{G} \to X/\mathcal{G}$ is dense minimal if and only if $\mathcal{T}_f$ is dense minimal.*

**Proof.** Suppose $F$ is dense minimal. Let $x \in X$ and let $U$ be a nonempty open subset of $X$. We see that $\mathcal{T}_f^n(x) \cap U \neq \emptyset$ for some $n \in \mathbb{N} \cup \{0\}$. Since $\mathcal{G}$ is a continuous decomposition, $q$ is open [15, Theorem 2.1.19]. Hence, $q(U)$ is a nonempty open subset of $X/\mathcal{G}$. Since $F$ is dense minimal, there exists $n \in \mathbb{N} \cup \{0\}$ such that $F^n(q(x)) \in q(U)$.

Note that $F^n(q(x)) = q(f^n(x))$, for each $x \in X$ and every $n \in \mathbb{N} \cup \{0\}$. Hence, $q(f^n(x)) \in q(U)$. Thus, $q^{-1}(q(f^n(x))) \cap U \neq \emptyset$. Since $q^{-1}(q(f^n(x))) = \mathcal{T}(\{f^n(x)\})$, we have that $q^{-1}(q(f^n(x))) \subseteq \mathcal{T}_f^n(x)$, Proposition 3.6. Therefore, $\mathcal{T}_f^n(x) \cap U \neq \emptyset$.

Assume $\mathcal{T}_f$ is dense minimal. Let $\mathcal{U}$ be an open subset of $X/\mathcal{G}$ and let $\chi \in X/\mathcal{G}$. Then there exists $x \in X$ such that $q^{-1}(\chi) = \mathcal{T}(\{x\})$ and $q^{-1}(\mathcal{U})$ is an open subset of $X$. Since $\mathcal{T}_f$ is dense minimal, $x$ has a dense orbit under $\mathcal{T}_f$. Hence, there exists $n \in \mathbb{N} \cup \{0\}$, such that $\mathcal{T}_f^n(x) \cap q^{-1}(\mathcal{U}) \neq \emptyset$. Thus, by Lemma 7.2, we have that $q^{-1}(F^n(\chi)) \cap q^{-1}(\mathcal{U}) \neq \emptyset$. Therefore, $F^n(\chi) \in \mathcal{U}$ and $F$ is dense minimal. $\square$

**7.4 Corollary.** *Let $X$ be a continuum such that $\mathcal{T}\colon 2^X \to 2^X$ is continuous. With Notation 7.1, $F\colon X/\mathcal{G} \to X/\mathcal{G}$ is dense minimal if and only if $\mathcal{T}_f$ is dense minimal.*

**Proof.** By [7, Theorem 3.7], $\mathcal{G} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ is a continuous decomposition. The corollary now follows from Theorem 7.3. $\square$

**7.5 Corollary.** *Let $X$ be a decomposable homogeneous continuum. With Notation 7.1. $F\colon X/\mathcal{G} \to X/\mathcal{G}$ is dense minimal if and only if $\mathcal{T}_f$ is dense minimal.*

**Proof.** By F. Burton Jones Aposyndetic Decomposition Theorem, [15, Theorem 5.1.18], $\mathcal{G} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ is a continuous decomposition. The corollary now follows from Theorem 7.3. □

**7.6 Remark.** Let $f_\theta \colon S^1 \to S^1$ be defined by $f_\theta(z) = ze^{i\pi\theta}$, for each $z \in S^1$. It is well-known that if $\theta \notin \mathbb{Q}$, then $f_\theta$ is minimal. By a theorem of W. Lewis [12, Theorem 4], there exists a homeomorphism $h \colon X \to X$, such that

$$f_\theta \circ q = q \circ h;$$

where $X$ is the 1-dimensional continuum (the circle of pseudo-arcs [5]), $q \colon X \to S^1$ is the quotient map, and $q^{-1}(z)$ is a terminal pseudo-arc. $X$ is a continuum such that $\mathcal{T}$ is continuous [15, Theorem 3.5.6]. Thus, $\mathcal{T}_h$ is dense minimal, Corollary 7.4. Note that, by the same argument, if there exists a minimal homeormorphism $h$ of the Menger curve of pseudo-arcs onto itself, then $\mathcal{T}_h$ would be dense minimal.

**7.7 Lemma.** *With Notation 7.1, suppose $f$ is onto. Then $\mathcal{G} = \{\mathcal{T}_f(x) \mid x \in X\}$.*

**Proof.** Let $\mathcal{G}' = \{\mathcal{T}_f(x) \mid x \in X\}$. Let $x$ be a point of $X$. Since $f$ is onto, there exists $z \in X$ such that $f(z) = x$. Hence, $\mathcal{T}(\{x\}) = \mathcal{T}(\{f(z)\}) = \mathcal{T}_f(z)$. Thus, $\mathcal{T}(\{x\}) \in \mathcal{G}'$ and $\mathcal{G} \subset \mathcal{G}'$.

Let $x$ be a point of $X$. Then $\mathcal{T}_f(x) = \mathcal{T}(\{f(x)\}) \in \mathcal{G}$. Hence, $\mathcal{G}' \subset \mathcal{G}$. Therefore, $\mathcal{G} = \mathcal{G}'$. □

**7.8 Theorem.** *With Notation 7.1, suppose $f$ is onto. Then $F$ is closed minimal if and only if $\mathcal{T}_f$ is closed minimal.*

**Proof.** Suppose $\mathcal{T}_f$ is not closed minimal. Then there exists a nonempty closed proper subset $A$ of $X$ such that $\mathcal{T}_f(A) \subset A$. This implies that $f(A) \subset A$ (Proposition 3.6). Note that $q(A)$ is a nonempty closed subset of $X/\mathcal{G}$ such that $F(q(A)) = q(f(A)) \subset q(A)$.

We prove that $q(A)$ is a proper subset of $X/\mathcal{G}$. Suppose this is not true, that is, $q(A) = X/\mathcal{G}$. Let $x$ be a point of $X$. Since $q(A) = X/\mathcal{G}$, we have that $A \cap \mathcal{T}(\{x\}) \neq \emptyset$. Let $z \in A \cap \mathcal{T}(\{x\})$. Since $\mathcal{G}$ is a decomposition, we have that $\mathcal{T}(\{z\}) = \mathcal{T}(\{x\})$. By [15, Remark 3.1.5], $x \in \mathcal{T}(\{x\})$. Hence, $f(x) \in f(\mathcal{T}(\{x\}))$. By Notation 7.1, there exists $w \in X$ such that $f(\mathcal{T}(\{x\})) \subset \mathcal{T}(\{w\})$. Since $f(x) \in f(\mathcal{T}(\{x\}))$ and $\mathcal{G}$ is a decomposition, we obtain that $\mathcal{T}(\{f(x)\}) = \mathcal{T}(\{w\})$. Thus, $\mathcal{T}_f(x) = \mathcal{T}(f(\{x\})) = \mathcal{T}(\{w\})$. Since $\mathcal{T}(\{z\}) = \mathcal{T}(\{x\})$, a similar argument shows that $\mathcal{T}_f(z) = \mathcal{T}(\{w\})$. Hence, $\mathcal{T}_f(x) = \mathcal{T}_f(z)$. Since $z \in A$, the fact that $\mathcal{T}_f(A) \subset A$ implies that $\mathcal{T}_f(z) \subset A$. Thus, $\mathcal{T}_f(x) \subset A$. Therefore, by Lemma 7.7, that $A = X$, a contradiction.

Conversely, suppose that $F$ is not closed minimal; i.e., there exists a proper closed subset $\mathcal{A}$ of $X/\mathcal{G}$ such that $F(\mathcal{A}) \subset \mathcal{A}$. Note that if $z \in \mathcal{T}_f(q^{-1}(\mathcal{A}))$, then there exists $x \in q^{-1}(\mathcal{A})$ such that $z \in \mathcal{T}_f(x)$, Corollary 3.3. Hence, $z \in \mathcal{T}(\{f(x)\})$ and $z \in q^{-1}(F(q(x)))$. Since $q(x) \in \mathcal{A}$, $q(z) = F(q(x))$, where $F(q(x)) \in \mathcal{A}$. Thus, $q(z) \in \mathcal{A}$ and $z \in q^{-1}(\mathcal{A})$. Therefore, $\mathcal{T}_f(q^{-1}(\mathcal{A})) \subset q^{-1}(\mathcal{A})$. Since $\mathcal{A} \neq X/\mathcal{G}$, there exists $\chi \in X/\mathcal{G} \setminus \mathcal{A}$. Hence, $q^{-1}(\chi) \cap q^{-1}(\mathcal{A}) = \emptyset$ and $q^{-1}(\mathcal{A})$ is a proper closed subset of $X$. Therefore, $\mathcal{T}_f$ is not closed minimal. □

## 8. $\mathcal{T}_{1_X}$ is dense minimal

Let

$$\mathfrak{D} = \{X \text{ is a continuum} \mid \mathcal{T}_{1_X} \text{ is dense minimal}\}.$$

A natural question is the following:

**8.1 Question.** What continua $X$ belong to $\mathfrak{D}$?

A continuum $X$ is said to be *$n$-indecomposable* provided that $X$ is the union of $n$ continua such that no one of them is a subset of the union of the others and $X$ is not the union of $n+1$ such continua.

A continuum $X$ is said to be *$\omega$-indecomposable* provided that $\mathcal{T}(\mathcal{F}_1(X))$ is a countably infinite set, where $\mathcal{F}_1(X) = \{\{x\} \mid x \in X\}$.

The next result follows from the definitions.

**8.2 Proposition.** *Let $X$ be a continuum. If $X$ is $n$-indecomposable, for some $n \in \mathbb{N}$, then $X \in \mathfrak{D}$.*

**Proof.** Suppose $X$ is an $n$-decomposable continuum, for some $n \in \mathbb{N}$. Then there exist $n$ indecomposable subcontinua $X$ such that $X = \bigcup_{j=1}^{n} M_j$. Let $x$ be a point of $X$. Then, by the proof of [6, Theorem 3.5], we have that $\mathcal{T}(\{x\}) = \bigcup\{M_j \mid x \in M_j\}$. If $\mathcal{T}(\{x\}) = X$, we are done. Assume that $\mathcal{T}(\{x\}) \neq X$. Hence, there exists $M_k \notin \{M_j \mid x \in M_j\}$ and $M_k \cap (\bigcup\{M_j \mid x \in M_j\}) \neq \emptyset$. Thus, since $M_k$ is an indecomposable continuum, $M_k \subset \mathcal{T}(\mathcal{T}(\{x\}))$ [15, Theorem 3.1.39]. Also, by the proof of [6, Theorem 3.5], $\mathcal{T}(\mathcal{T}(\{x\})) = \bigcup\{M_j \mid \mathcal{T}(\{x\}) \cap M_j \neq \emptyset\}$. If $\mathcal{T}(\mathcal{T}(\{x\})) = X$, we are done. If $\mathcal{T}(\mathcal{T}(\{x\})) \neq X$, repeat the previous argument. Hence, there exists $m \in \mathbb{N}$ $(m \leq n)$ such that $\mathcal{T}^m(\{x\}) = X$. Therefore, $X \in \mathfrak{D}$. $\square$

Let $X$ and $Y$ be the continua defined in Example 5.8. Note that $X$ and $Y$ are $\omega$-indecomposable continua, $Y \in \mathfrak{D}$, but $X$ does not belong to $\mathfrak{D}$.

**8.3 Question.** Is there a hereditarily decomposable continuum in $\mathfrak{D}$?

By Corollary 10.6, if $\mathcal{T}_{1_X}$ is dense minimal and $X$ is irreducible, then $X$ contains an indecomposable subcontinuum with nonempty interior. Thus, there is not an irreducible hereditarily decomposable continuum in $\mathfrak{D}$.

## 9. Sensitivity

**9.1 Lemma.** *Let $X$ be a continuum. If $f\colon X \to X$ is an exact map, then $f$ is sensitive.*

**Proof.** Let $\varepsilon = \operatorname{diam}(X)/2$. Let $x_0 \in X$ and let $\delta > 0$. Since $f$ is exact, there exists $n_0 \in \mathbb{N}$ such that $f^{n_0}(\mathcal{V}_\delta(x_0)) = X$. Hence, there exist $z, w \in \mathcal{V}_\delta(x_0)$ such that $d(f^{n_0}(z), f^{n_0}(w)) = \operatorname{diam}(X)$. Also, $d(f^{n_0}(w), f^{n_0}(z)) \leq d(f^{n_0}(w), f^{n_0}(x_0)) + d(f^{n_0}(x_0), f^{n_0}(z))$. Therefore, either $d(f^{n_0}(w), f^{n_0}(x_0)) \geq \varepsilon$ or $d(f^{n_0}(x_0), f^{n_0}(z)) \geq \varepsilon$. $\square$

**9.2 Example.** Let $G, H\colon F_\mathcal{C} \to F_\mathcal{C}$ be the maps given in Example 5.4. Note that in part (3) of that example, we have that $\mathcal{T}_G$ and $\mathcal{T}_H$ are exact and neither $\mathcal{T}_G$ nor $\mathcal{T}_H$ is sensitive.

**9.3 Proposition.** *There exist a continuum $X$ and a sensitive map $\varphi\colon X \to X$ such that $\mathcal{T}_\varphi$ is not sensitive.*

**Proof.** Let $f, g\colon [0,1] \to [0,1]$ be defined by

$$g(t) = \begin{cases} 3t, & \text{if } 0 \leq t \leq \frac{1}{3}; \\ 2-3t, & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}; \\ 3t-2, & \text{if } \frac{2}{3} \leq t \leq 1, \end{cases}$$

and

$$f(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq \frac{1}{2}; \\ 2-2t, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

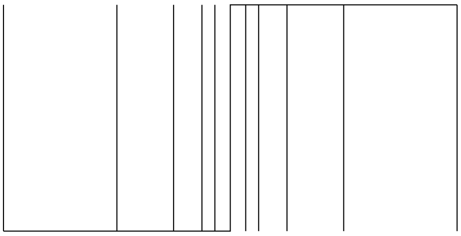

**Fig. 4.** $\varphi$ sensitive.

Note that $f$ and $g$ are exact. Hence, they are sensitive, Lemma 9.1.

Let $X = (\{0\} \times [0,1]) \cup \left[\bigcup_{n\in\mathbb{N}}(\{\frac{1}{2^n}\} \times [0,1])\right] \cup \left[\bigcup_{n\in\mathbb{N}}(\{-\frac{1}{2^n}\} \times [0,1])\right] \cup ([-1,0] \times \{0\}) \cup ([0,1] \times \{1\})$ and let $\varphi\colon X \to X$ be given by

$$\varphi(t,s) = \begin{cases} (f(t), g(s)), & \text{if } t \geq 0; \\ (-f(-t), g(s)), & \text{if } t \leq 0. \end{cases}$$

Then $\varphi$ is sensitive (see Fig. 4). Also, since $\mathcal{T}_\varphi(z) = \{0\} \times [0,1]$ for each $z \in \{1\} \times [0,1]$ and $\mathcal{T}_\varphi(\{0\} \times [0,1]) = \{0\} \times [0,1]$, we obtain that $\mathcal{T}_\varphi$ is not sensitive. □

## 10. Irreducible continua

A continuum $X$ is *irreducible* if there exist two points $p$ and $q$ of $X$ such that no proper subcontinuum of $X$ contains both $p$ and $q$. A continuum $X$ is of *type* $\lambda$ provided that $X$ is irreducible and each indecomposable subcontinuum of $X$ has empty interior. By [17, Theorem 10, p. 15], a continuum $X$ is of type $\lambda$ if and only if it admits a finest monotone upper semicontinuous decomposition $\mathcal{G}$ such that $X/\mathcal{G}$ is an arc. Each element of $\mathcal{G}$ is called a *layer* of $X$.

**10.1 Theorem.** *Let $X$ be irreducible between $p$ and $q$. Then $\mathcal{T}_{1_X}$ is exact if and only if there exists $n \in \mathbb{N}$ such that $X$ is $n$-indecomposable.*

**Proof.** Suppose first that $\mathcal{T}_{1_X}$ is exact. Since our theorem is true for an indecomposable continuum, we assume that $X$ is decomposable. Let $A$ and $B$ be proper subcontinua such that $X = A \cup B$. Let $p \in A \setminus B$ and let $q \in B \setminus A$. We show that $\mathcal{T}(\{p\})$ has nonempty interior. Suppose that $Int_X(\mathcal{T}(\{p\})) = \emptyset$. Note that $U = X \setminus A$ is an open connected subset of $X$ such that $q \in U$ [15, Theorem 1.7.31]. Hence, $Cl_X(U)$ is a proper subcontinuum of $X$. Since $Int_X(\mathcal{T}(\{p\})) = \emptyset$, $\mathcal{T}(\{p\}) \cap Cl_X(U) = \emptyset$. Then for each $x \in Cl_X(U)$, there exists a continuum $L_x$ such that $x \in Int_X(L_x) \subset L_x \subset X \setminus \{p\}$. Since $Cl_X(U)$ is compact, there exist $x_1, \ldots, x_n$ in $Cl_X(U)$ such that

$$Cl_X(U) \subset Int_X(\bigcup_{i=1}^{n} L_{x_i}) \subset \bigcup_{i=1}^{n} L_{x_i} \subset X \setminus \{p\}.$$

Note that $R_1 = \bigcup_{i=1}^{n} L_{x_i}$ is a continuum with nonempty interior and $p \notin R_1$. With the same argument, $\mathcal{T}(\{p\}) \cap R_1 = \emptyset$ and there exists a subcontinuum $R_2$ of $X$ such that

$$R_1 \subset Int_X(R_2) \subseteq R_2 \subseteq X \setminus \{p\}.$$

Since $Cl_X(U) \subset Int_X(R_1)$ and $Cl_X(X \setminus R_1)$ is a continuum [15, Theorem 1.7.31] such that $Cl_X(X \setminus R_1) \cap Cl_X(U) = \emptyset$, we have that $\mathcal{T}_{1_X}(Cl_X(U)) \cap X \setminus R_1 = \emptyset$. Thus, $\mathcal{T}_{1_X}(Cl_X(U)) \subset R_1$. Continuing with this process, for each $m \in \mathbb{N}$, there exists a subcontinuum $R_m$ of $X$ such that $R_i \subset Int_X(R_{i+1}) \subset R_{i+1} \subset X \setminus \{p\}$ and $\mathcal{T}_{1_X}(R_i) \subset R_{i+1}$ for every $i \in \{1, \dots, m-1\}$.

Thus,

$$\mathcal{T}_{1_X}^i(U) \subset \mathcal{T}_{1_X}^i(Cl_X(U)) = \mathcal{T}_{1_X}^{i-1}(\mathcal{T}_{1_X}(Cl_X(U))) \subset \mathcal{T}_{1_X}^{i-1}(R_1) \subset \cdots \subset R_i \subset X \setminus \{p\}.$$

Hence, $\mathcal{T}_{1_X}^n(U) \neq X$ for each $n \in \mathbb{N}$ and $\mathcal{T}_{1_X}$ is not exact. Therefore, $\mathcal{T}(\{p\})$ has nonempty interior.

Let $V = Int_X(\mathcal{T}(\{p\}))$. Since $\mathcal{T}_{1_X}$ is exact, there exists $k \in \mathbb{N}$ such that $\mathcal{T}_{1_X}^k(V) = X$. Since $\mathcal{T}_{1_X}^k(V) \subset \mathcal{T}_{1_X}^{k+1}(p)$, $\mathcal{T}_{1_X}^{k+1}(p) = X$. Therefore, by [8, Theorem 4], $X$ is $n$-indecomposable for some $n \in \mathbb{N}$.

The converse implication follows from the definition of $n$-indecomposable continuum. □

We have the following result from Remark 4.4 and Theorems 5.19 and 10.1.

**10.2 Corollary.** *Let $X$ be an irreducible continuum. Then the following are equivalent:*

(1) *$X$ is $n$-indecomposable, for some $n \in \mathbb{N}$;*
(2) *$\mathcal{T}_{1_X}$ is exact;*
(3) *$\mathcal{T}_f$ is exact, for each homeomorphism $f\colon X \to X$;*
(4) *$\mathcal{T}_{1_X}$ is strongly transitive.*

The following example shows that the irreducibility of $X$ cannot be omitted in Corollary 10.2,

**10.3 Example.** There exists a continuum $X$ such that $\mathcal{T}_{1_X}$ is exact, but $X$ is not $n$-indecomposable, for any $n \in \mathbb{N}$.

Let $W$ be an indecomposable continuum and let $p \in W$. Let $Z = (\{p\} \times [0,1]) \cup (\bigcup_{i=1}^{\infty}(W \times \{\frac{1}{n}\})) \cup (W \times \{0\})$. Note that $Z$ is a continuum. Let $X = Z/(\{p\} \times [0,1])$ and let $v$ be the point obtained from $\{p\} \times [0,1]$, in the quotient space. Observe that $\mathcal{T}_{1_X}(v) = X$ and $v \in \mathcal{T}_{1_X}(x)$ for each $x \in X$. Thus, $\mathcal{T}_{1_X}^2(x) = X$ and $\mathcal{T}_{1_X}$ is exact.

**10.4 Theorem.** *Let $X$ be an irreducible continuum. If $\mathcal{T}_{1_X}$ is transitive, then $X$ contains an indecomposable subcontinuum with nonempty interior.*

**Proof.** Suppose to the contrary that every subcontinuum with nonempty interior is decomposable. Hence, there exists a monotone map $f\colon X \to [0,1]$, [17, Theorem 10]. Now, the theorem follows from Theorem 4.6. □

Theorem 10.4 can be written as follows:

**10.5 Corollary.** *If $X$ is a type $\lambda$ continuum, then $\mathcal{T}_{1_X}$ is not transitive.*

The following corollary follows from Theorem 10.4 and Diagram 2.

**10.6 Corollary.** *Let $X$ be an irreducible continuum. If $\mathcal{T}_{1_X}$ is exact, mixing, totally transitive, strongly transitive, backward dense minimal or dense minimal, then $X$ contains an indecomposable subcontinuum with nonempty interior.*

**10.7 Lemma.** *Let $X$ be an irreducible continuum between $p$ and $q$ such that $\mathcal{T}_{1_X}$ is transitive. If $U_p$ and $U_q$ are open connected subsets of $X$ such that $p \in U_p$, $q \in U_q$ and $Cl_X(U_p) \cap Cl_X(U_q) = \emptyset$, then there exist indecomposable continua $S_0, \dots, S_l$ such that:*

(1) *$Int_X(S_i) \neq \emptyset$ for each $i \in \{0, \dots, l\}$;*
(2) *$X = Cl_X(U_p) \cup S_l \cup \cdots \cup S_0 \cup Cl_X(U_q)$;*
(3) *$|\{S_i \mid x \in S_i\}| \leq 2$ for each $x \in X \setminus (Cl_X(U_p) \cup Cl_X(U_q))$.*
(4) *$\mathcal{T}(\{x\}) = \bigcup\{S_i \mid x \in S_i\}$ for each $x \in X \setminus (Cl_X(U_p) \cup Cl_X(U_q))$.*

**Proof.** Since $\mathcal{T}_{1_X}$ is transitive, there exists $k_0 \in \mathbb{N}$ such that $\mathcal{T}_{1_X}^{k_0}(Cl_X(U_p)) \cap Cl_X(U_q) \neq \emptyset$ and $\mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cap Cl_X(U_q) = \emptyset$. Hence, there exists $w_0 \in \mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p))$ such that $\mathcal{T}(\{w_0\}) \cap Cl_X(U_q) \neq \emptyset$. Since $X$ is irreducible, $X = \mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cup \mathcal{T}(\{w_0\}) \cup Cl_X(U_q)$. Let $S_0$ be an irreducible subcontinuum of $\mathcal{T}(\{w_0\})$ between $\mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p))$ and $Cl_X(U_q)$. Thus, $X = \mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cup S_0 \cup Cl_X(U_q)$. Note that $X \setminus (\mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cup Cl_X(U_q))$ is a nonempty open subset of $S_0$. Hence, $Int_X(S_0) \neq \emptyset$.

We see that $S_0$ is indecomposable. Suppose that $S_0 = A \cup B$, where $A$ and $B$ are proper subcontinua of $S_0$. Since $X$ is irreducible, we may assume that:

- $\mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cap A \neq \emptyset$ and $B \cap Cl_X(U_q) \neq \emptyset$; and
- $(\mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cup A) \cap Cl_X(U_q) = \emptyset$ and $\mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cap (B \cup Cl_X(U_q)) = \emptyset$.

Note that $V = X \setminus (\mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cup A \cup Cl_X(U_q))$ is a nonempty open subset of $B$. Also, $w_0 \in \mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p))$ and $V \subset \mathcal{T}(\{w_0\})$. If $z \in V$, then $z \in Int_X(B)$ and $w_0 \notin B$. This contradicts that $z \in \mathcal{T}(\{w_0\})$. Thus, $S_0$ is indecomposable.

Now, if $S_0 \cap Cl_X(U_p) \neq \emptyset$, then $X = Cl_X(U_p) \cup S_0 \cup Cl_X(U_q)$. Hence, we have *2*. Suppose that $S_0 \cap Cl_X(U_p) = \emptyset$. Since $\mathcal{T}_{1_X}^{k_0-1}(Cl_X(U_p)) \cap (S_0 \cup Cl_X(U_q)) \neq \emptyset$, there exists $k_1 < k_0$ such that $\mathcal{T}_{1_X}^{k_1}(Cl_X(U_p)) \cap S_0 \neq \emptyset$ and $\mathcal{T}_{1_X}^{k_1-1}(Cl_X(U_p)) \cap S_0 = \emptyset$. Let $w_1 \in \mathcal{T}_{1_X}^{k_1-1}(Cl_X(U_p))$ be such that $\mathcal{T}(\{w_1\}) \cap S_0 \neq \emptyset$. Since $X$ is irreducible, $X = \mathcal{T}_{1_X}^{k_1-1}(Cl_X(U_p)) \cup \mathcal{T}(\{w_1\}) \cup S_0 \cup Cl_X(U_q)$. Let $S_1$ be an irreducible subcontinuum of $\mathcal{T}(\{w_1\})$ between $\mathcal{T}_{1_X}^{k_1-1}(Cl_X(U_p))$ and $S_0$. Thus,

$$X \setminus (\mathcal{T}_{1_X}^{k_1-1}(Cl_X(U_p)) \cup S_0 \cup Cl_X(U_q)) \subset S_1$$

and $X = \mathcal{T}_{1_X}^{k_1-1}(Cl_X(U_p)) \cup S_1 \cup S_0 \cup Cl_X(U_q)$. With a similar argument, we show that $S_1$ is indecomposable. If $Cl_X(U_p) \cap S_1 \neq \emptyset$, then $X = Cl_X(U_p) \cup S_1 \cup S_0 \cup Cl_X(U_q)$. Otherwise, inductively, there exist indecomposable continua $S_0, \dots, S_l$ such that $X = Cl_X(U_p) \cup S_l \cup \cdots \cup S_1 \cup S_0 \cup Cl_X(U_q)$ for some $l \leq k_0$. We have proved *2*. Also, by construction, $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| \leq 1$ for each $i, j \in \{0, \dots, l\}$. Thus, we obtain *3*.

Note that *4* is clear. □

**10.8 Theorem.** *Let $X$ be an irreducible continuum between $p$ and $q$ such that $\mathcal{T}_{1_X}$ is transitive. Then:*

(1) *If there exist indecomposable continua with nonempty interior containing $p$ and $q$, then $X$ is $n$-indecomposable for some $n \in \mathbb{N}$.*
(2) *If either $p$ or $q$ is not contained in an indecomposable continuum with nonempty interior, then $X$ is an $\omega$-indecomposable continuum.*

**Proof.** If $X$ is indecomposable, we have *1*. Let $A$ and $B$ be indecomposable subcontinua with nonempty interior of $X$ such that $p \in A$ and $q \in B$. If $A \cap B \neq \emptyset$, $X = A \cup B$ is 2-indecomposable, because $X$ is irreducible. Thus, we assume that $A \cap B = \emptyset$. Note that $W = X \setminus (A \cup B)$ is an open connected subset of $X$ [11, Theorem 4, p. 193]. Also, $Cl_X(X \setminus (Cl_X(W) \cup B))$ is a subcontinuum of $A$ such that $Int_X(Cl_X(X \setminus (Cl_X(W) \cup B))) \neq \emptyset$. Hence, $Cl_X(X \setminus (Cl_X(W) \cup B)) = A$ because $A$ is indecomposable. Similarly, we show that $B = Cl_X(X \setminus (Cl_X(W) \cup A))$. By Lemma 10.7, $X = A \cup S_l \cup \cdots \cup S_0 \cup B$, where

$S_0, \dots, S_l$ are indecomposable subcontinua of $X$. Since $X$ is irreducible, $X$ is $(l+3)$-indecomposable. We have proved *1*.

Suppose that $p$ does not belong to any indecomposable continuum with nonempty interior and there exists an indecomposable continuum $L$ such that $q \in L$ and $Int_X(L) \neq \emptyset$. Note that $A_1 = Cl_X(X \setminus L)$ is a continuum with nonempty interior, and $L = Cl_X(X \setminus A_1)$. Since $p \in A_1$, $A_1$ is decomposable. Let $B_1$ and $B_2$ be proper subcontinua of $A_1$ such that $A_1 = B_1 \cup B_2$. Suppose that $p \in B_1$. Let $A_2 = Cl_X(X \setminus (B_2 \cup L))$. Observe that $A_2 \cap L = \emptyset$. Thus, there exist indecomposable continua $S_1, \dots, S_{n_1}$ such that $X = A_2 \cup S_{n_1} \cup \dots \cup S_1 \cup L$, Lemma 10.7. Since $A_2$ has nonempty interior and $p \in A_2$, we have that $A_2$ is decomposable. Hence, there exist proper subcontinua $C_1$ and $C_2$ of $A_2$ such that $A_2 = C_1 \cup C_2$. We suppose that $p \in C_1$. Let $A_3 = Cl_X(X \setminus (C_2 \cup S_{n_1} \cup \dots \cup S_1 \cup L))$. Since $A_3 \cap (S_{n_1} \cup \dots \cup S_1 \cup L) = \emptyset$, by Lemma 10.7, there exist indecomposable continua $S_{n_1+1}, \dots, S_{n_2}$ such that $X = A_3 \cup S_{n_2} \cup \dots \cup S_1 \cup L$, for some $n_2 > n_1$. Inductively, there exists a sequence of continua $\{A_n\}_{n \in \mathbb{N}}$ such that:

- $p \in Int_X(A_i)$ for each $i \in \mathbb{N}$;
- $A_{i+1} \subset A_i$ for each $i \in \mathbb{N}$;
- $X = A_{i+1} \cup S_{n_i} \cup \dots \cup S_1 \cup L$, where $S_1, \dots, S_{n_i}$ are indecomposable continua with nonempty interior, and $n_i < n_{i+1}$ for each $i \in \mathbb{N}$.

Thus,

$$X = \left( \bigcap_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} S_n \right) \cup L,$$

where:

- $L \cap S_i \neq \emptyset$ if and only if $i = 1$;
- $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| \leq 1$;
- $p \in \bigcap_{n=1}^{\infty} A_n$ and $(\bigcap_{n=1}^{\infty} A_n) \cap ((\bigcup_{n=1}^{\infty} S_n) \cup L) = \emptyset$.

Let $K = \bigcap_{n=1}^{\infty} A_n$. We show that $\mathcal{T}(\{x\}) = K$ for each $x \in K$. Let $x_0 \in K$. Let $z \in X \setminus K$. Then there exists $k_0 \in \mathbb{N}$ such that $z \in Int_X(S_{k_0} \cup \dots \cup S_1 \cup L)$. Note that $x_0 \notin S_{k_0} \cup \dots \cup S_1 \cup L$ and $S_{k_0} \cup \dots \cup S_1 \cup L$ is a continuum. Thus, $z \notin \mathcal{T}(\{x_0\})$ and $\mathcal{T}(\{x_0\}) \subset K$. This implies, by the transitivity of the function $\mathcal{T}_{1_X}$, that $Int_X(K) = \emptyset$. Also, we have that $K \subset Cl_X(\bigcup_{i=1}^{\infty} S_i) \cup L$, because $Cl_X(\bigcup_{i=1}^{\infty} S_i) \cup L$ is a continuum and $X \setminus (Cl_X(\bigcup_{i=1}^{\infty} S_i) \cup L) \subset K$.

Now, let $z \in K$ and let $P$ be a continuum such that $z \in Int_X(P)$. Since $Int_X(K) = \emptyset$, there exists $l \in \mathbb{N}$ such that $P \cap S_l \neq \emptyset$. Hence, $X = K \cup P \cup (\bigcup_{i=1}^{l} S_i) \cup L$. Thus, $\bigcup_{i=l+1}^{\infty} S_i \subset P$ and, since $P$ is compact, $Cl_X(\bigcup_{i=l+1}^{\infty} S_i) \subset P$. Since $Cl_X(\bigcup_{i=1}^{\infty} S_i) \cup L = Cl_X(\bigcup_{i=l+1}^{\infty} S_i) \cup S_l \cup \dots \cup S_1 \cup L$ and $K \cap (S_l \cup \dots \cup S_1 \cup L) = \emptyset$, we have that $K \subset Cl_X(\bigcup_{i=l+1}^{\infty} S_i) \subset P$. Hence, $x_0 \in P$ and $z \in \mathcal{T}(\{x_0\})$. Therefore, $\mathcal{T}(\{x\}) = K$ for each $x \in K$.

Now, if $x \in X$, then

$$\mathcal{T}(\{x\}) = \begin{cases} K, & \text{if } x \in K; \\ S_i, & \text{if } x \in S_i \setminus (K \cup (\bigcup_{n \neq i} S_n) \cup L); \\ S_i \cup S_{i+1}, & \text{if } x \in S_i \cap S_{i+1},\ i \in \mathbb{N}; \\ L \cup S_1, & \text{if } x \in L \cap S_1; \\ L, & \text{if } x \in L \setminus S_1. \end{cases}$$

Therefore, $\mathcal{T}(\mathcal{F}_1(X))$ is countable, and $X$ is $\omega$-indecomposable. If neither $p$ nor $q$ belongs to an indecomposable continuum with nonempty interior, then a similar argument shows that $\mathcal{T}(\mathcal{F}_1(X))$ is a countable set. □

The next result follows from Remark 4.3 and Theorem 10.8.

**10.9 Corollary.** *Let $X$ be an irreducible continuum. If $\mathcal{T}_{1_X}$ is either mixing or totally transitive, then:*

(1) *If there exist indecomposable continua with nonempty interior containing $p$ and $q$, then $X$ is $n$-indecomposable for some $n \in \mathbb{N}$.*
(2) *If either $p$ or $q$ is not contained in an indecomposable continuum with nonempty interior, then $X$ is an $\omega$-indecomposable continuum.*

## Acknowledgement

The authors thank the referee for the valuable suggestion made that improved the paper. The authors also thank Professor H. Kato for the useful conversations he had with the first named author. The first named author thanks the financial support given for the Project 2459 to *La Vicerrectoría de Investigación y Extensión de la Universidad Industrial de Santander y su Programa de Movilidad*. The second and third named authors thank the *Universidad Industrial de Santander, Colombia*, for the support given during this research.

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