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The hyperspace of closed subsets on the edge of irreducible continua

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Abstract

For a nonempty nowhere dense closed subset of a continuum X, consider the following properties: being a non-weak cut subset, a non-block subset, a weak non-block subset, a shore subset, a non-strong center, and a non-cut subset of X. In this paper, we provide necessary conditions for subsets of an irreducible continuum about a subset to have one of these properties, and we prove that these properties are equivalent for nonempty nowhere dense closed subsets of an irreducible continuum about a finite subset. This result completes the study previously conducted on non-cut points and for subsets on the edge of a continuum by several authors.

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1. Introduction

A continuum is a nondegenerate compact connected metric space. A nonempty compact connected subspace of a metric space is called *subcontinuum*. A nonempty subset L of a metric space Z is *continuum-wise connected* provided that for each $x, y \in L$, there exists a subcontinuum K of Z such that $x, y \in K \subseteq L$. The maximal continuum-wise connected subsets of a metric space are called

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continuum-wise connected component. A metric space is *continuum-wise connected* if it possesses a unique continuum-wise connected component.

A nonempty nowhere dense closed subset B of a continuum X is a

- *non-weak cut subset of* X[1] if $X \setminus B$ is continuum-wise connected.
- *non-block subset of* X if each continuum-wise connected component of $X \setminus B$ is a dense subset of X.
- *weak non-block subset* of X if a continuum-wise connected component of the subspace $X \setminus B$ is a dense subset of X.
- *shore subset of* X if for each $\varepsilon > 0$, there exists a subcontinuum L of $X \setminus B$ such that $H(L, X) < \varepsilon$ (H denotes the Hausdorff metric defined in [13, p. 11]).
- *not a strong center of* X if for each pair of nonempty open subsets U and V of X, there exists a subcontinuum M of $X \setminus B$ such that $U \cap M \neq \emptyset$ and $V \cap M \neq \emptyset$.
- *non-cut subset of* X if $X \setminus B$ is connected.

The last concepts are related to the idea of *being on the edge* of a continuum or belonging of a kind of non-cut subset of a continuum. They have been studied in various contexts in [1, 2, 3, 4, 5, 7, 8, 10, 12, 15, 16].

For a continuum X, let

$$\begin{aligned}
2^X &= \{B \subseteq X : B \text{ is a nonempty closed subset of } X\},\\
NWC(X) &= \{B \in 2^X : B \text{ is a non-weak cut subset of } X\},\\
NB(X) &= \{B \in 2^X : B \text{ is a non-block subset of } X\},\\
NB^*(X) &= \{B \in 2^X : B \text{ is a weak non-block subset of } X\},\\
S(X) &= \{B \in 2^X : B \text{ is a shore subset of } X\},\\
NSC(X) &= \{B \in 2^X : B \text{ is not a strong center of } X\}, \text{ and}\\
NC(X) &= \{B \in 2^X : B \text{ is a non-cut subset of } X\}.
\end{aligned}$$

These subsets of 2^X are called *hyperspaces of subsets on the edge of* X[2]. The following result describes the relationship among the hyperspaces of subsets on the edge of a continuum and its proof follows from [8, Theorem 3.2, p. 99] and [7, Theorem 2.1, p. 3] together.

Theorem 1.1. *For a continuum X, the inclusions*

$$NWC(X) \subseteq NB(X) \subseteq NB^*(X) \subseteq S(X) \subseteq NSC(X) \subseteq NC(X)$$

holds.

[1] also called *strong non-cut subset* of X, see for instance [1, p. 2]

[2] also called *hyperspaces of non-cut subsets of* X, see for instance [1, p. 2]

The aim of [3, 7, 8] is to present characterizations of certain classes of continua using these hyperspaces and the relationships between each pair of them. In this paper, we contribute to completing that study for irreducible continua about a subset by improving some of the results presented in [7, 8, 14] and providing a positive answer to [8, Question 6.13, p. 106], and we also use the property of Kelley to establish equivalences among hyperspaces of subsets on the edge of a continuum.

2. Preliminaries and auxiliary results

The classical notion of irreducibility for a continuum X refers to the existence of a pair of points $p, q \in X$ such that no proper subcontinuum of X contains both. In [9], the author presents the following notion of irreducibility.

The symbol $X = irr(A)$ means that X is a continuum and A is a nonempty subset of X such that no proper subcontinua of X contains A, and for each $a \in A$, there exists a proper subcontinuum of X containing $A \setminus \{a\}$. A continuum X is *irreducible about* a subset A of X if $X = irr(A)$[3].

See [9, Proposition 7.2.7, p. 137] for the proof of the following result.

Proposition 2.1. *If $X = irr(A)$, then A is at most countable.*

Let A be a nondegenerate subset of a continuum X. For each $a \in A$, set

$$\lambda(a, A) = X \setminus \left\{ x \in X : \begin{array}{c} \text{there exists a proper subcontinuum} \\ K \text{ of } X \text{ such that } \{x\} \cup (A \setminus \{a\}) \subseteq K \end{array} \right\}.$$

The development of the following result draws on the proofs of [9, Propositions 7.2.9, 7.2.11 and 7.2.12, pp. 138, 139]. Compare with [17, 5.20, Theorem 11.4, pp. 83 and 198].

Proposition 2.2. *If $X = irr(A)$ and $a \in A$, then each of the following conditions holds.*

(1) *$a \in \lambda(a, A)$.*
(2) *$\lambda(a, A)$ is a connected subset of X.*
(3) *$X \setminus \lambda(a, A)$ is a connected dense subset of X.*
(4) *If C is a subcontinuum of X such that $\lambda(a, A) \cap C \neq \emptyset$ and $C \setminus \lambda(a, A) \neq \emptyset$, then $\lambda(a, A)$ is a subset of the interior of C.*

Lemma 2.3. *If $X = irr(A)$, $a \in A$ and $x \in \lambda(a, A)$, then $X = irr(\{x\} \cup (A \setminus \{a\}))$.*

Proof. First, from the definition of $\lambda(a, A)$, it follows that each proper subcontinuum of X containing $A \setminus \{a\}$ is disjoint from $\lambda(a, A)$. Hence, no proper subcontinua of X contains $\{x\} \cup (A \setminus \{a\})$.

[3] also called *strongly irreducible continuum about a subset*, see for instance [8, p. 98]

Now, let C be a proper subcontinuum of X containing $A \setminus \{a\}$. Then $C \cap \lambda(a, A) = \emptyset$. This guarantees that $(\{x\} \cup (A \setminus \{a\})) \setminus \{x\}$ is a subset of C.

Next, let $b \in A \setminus \{a\}$. Then there exists a proper subcontinuum D of X containing $A \setminus \{b\}$. Since $a \in A \setminus \{b\}$, a is an element of $D \cap \lambda(a, A)$. If $D \setminus \lambda(a, A) \neq \emptyset$, then $x \in \lambda(a, A) \subseteq D$ (see (4) of Proposition 2.2). Thus, D is a proper subcontinuum of X containing $(\{x\} \cup (A \setminus \{a\})) \setminus \{b\}$. Assume that $D \subseteq \lambda(a, A)$. If p were a point of $A \setminus \{a, b\}$, then p would be a point of D and each proper subcontinuum M of X containing $A \setminus \{a\}$ would satisfy $p \in M \subseteq X \setminus \lambda(a, A) \subseteq X \setminus D$. A contradiction. This implies that $A = \{a, b\}$. So, $\{x\}$ is a proper subcontinuum of X such that $x \in \{x\} = (\{x\} \cup (A \setminus \{a\})) \setminus \{b\}$.

Therefore, $X = irr(\{x\} \cup (A \setminus \{a\}))$. □

The notation $X = u - irr(A)$ means that $X = irr(A)$ and if E is a subset of X such that $X = irr(E)$, then $A = E$.

Theorem 2.4. *If $X = irr(A)$, then $X = u - irr(A)$ if and only if $\lambda(a, A) = \{a\}$ for each $a \in A$.*

Proof. First, assume that $X = u - irr(A)$. Let $a \in A$ and let $x \in \lambda(a, A)$. Lemma 2.3 ensures that $X = irr(\{x\} \cup (A \setminus \{a\}))$. Then $A = \{x\} \cup (A \setminus \{a\})$. This implies that $x = a$. Therefore, $\lambda(a, A) = \{a\}$.

Now, assume that $\lambda(a, A) = \{a\}$ for each $a \in A$ and let B be a subset of X such that $X = irr(B)$. By [9, Theorem 7.2.18, p. 140], there exists a bijection $f : A \to B$ such that $\lambda(a, A) = \lambda(f(a), B)$. Since $f(a) \in \lambda(f(a), B)$ and $\lambda(a, A) = \{a\}$ for each $a \in A$, f is the identity function. So, $A = B$. □

The notation $X = c - irr(A)$ means that $X = irr(A)$ and $\lambda(a, A)$ is a compact subspace of X for each $a \in A$. So, if $X = c - irr(A)$ and $a \in A$, then $\lambda(a, A)$ is a subcontinuum of X.

If $X = irr(A)$, the symbol $\eta(A)$ denotes the set $X \setminus \bigcup_{a \in A} \lambda(a, A)$.

Recall that a Baire space is a topological space in which the intersection of any countable family of open dense subsets is dense. Baire theorem [18, Theorem 25.3, p. 186] states that every compact Hausdorff space is a Baire space. Hence, each continuum is a Baire space.

Theorem 2.5. *If $X = c - irr(A)$, then $\eta(A)$ is a continuum-wise connected dense subset of X.*

Proof. The set $\eta(A)$ is continuum-wise connected (see [9, Theorem 7.3.9, p. 147]). Now, use Proposition 2.1 to conclude that $\{X \setminus \lambda(a, A) : a \in A\}$ is a countable family of dense open subsets of X. Since X is a Baire space, $\bigcap_{a \in A}(X \setminus \lambda(a, A))$ is a dense subset of X. Therefore, $\eta(A)$ is a dense subset of X. □

The following result is proven in [9, Proposition 7.3.10, p. 148].

Proposition 2.6. *If $X = c - irr(A)$, $x \in \eta(A)$ and $p \in A$, then there exists a subcontinuum Y of X such that $x \in Y$ and $Y \cap A = \{p\}$.*

Note that the collection of continuum-wise connected components of a metric space of Z is a partition finer than the collection of connected components of Z.

Let X be a continuum. The hyperspace $C(X)$ is the space consisting of all connected members of 2^X topologized by the Hausdorff metric. From [17, Theorem 4.5, p. 54], it follows that for each open subset U of X, the sets $U^+ = \{B \in C(X) : B \subseteq U\}$ and $U^- = \{B \in C(X) : B \cap U \neq \emptyset\}$ are open subsets of $C(X)$.

Let X be a continuum and let $p \in X$. The continuum X *has the property of Kelley at* p if for each element N of $C(X)$ containing p and for each open subset $\mathcal{U}$ of $C(X)$ containing N, there exists an open subset V of X containing p such that each point of V belongs to an element of $\mathcal{U}$. A continuum *has the property of Kelley* if it has the property of Kelley at each of its points.

Recall that a continuum X is *connected in kleinen at* $p \in X$, written X is c.i.k. at p, if for each open subset U of X containing p, there exists a subcontinuum L of X such that p is an element of the interior of L and U contains L.

Lemma 2.7. *Let X be a continuum having the property of Kelley. If K is a continuum-wise connected component of an open subset U of X and X is c.i.k. at some point of K, then K is an open subset of X.*

Proof. Choose $p \in K$ satisfying X is c.i.k. at p and a subcontinuum W of X contained in U such that p is an interior point of W. The interior of W is denoted by $\mathrm{Int}(W)$. Notice that W is a subset of K. Let $z \in K$. Take a subcontinuum Y of X such that $p, z \in Y \subseteq K$. Hence, $Y \in U^+ \cap (\mathrm{Int}(W))^-$. Since X has the property of Kelley at z, there exists an open subset V of X containing z such that each element of V belongs to an element of the open subset $U^+ \cap (\mathrm{Int}(W))^-$ of $C(X)$. Let us prove that V is a subset of K. Let $y \in V$. Then there exists $L \in U^+ \cap (\mathrm{Int}(W))^-$ containing y. This means L is a subset of U and L intersects W. Thus, $L \cup W$ is a subcontinuum of X containing both p and y, and contained in U. Then $L \cup W$ is a subset of K. Therefore, z is an interior point of K. In conclusion, K is an open subset of X. □

3. Main results

The following result strengthens [8, Corollary 3.7, p. 100] which asserts that $NB^*(X)$ contains at least two elements for any continuum.

Theorem 3.1. *The hyperspace $NB^*(X)$ contains at least three elements for any continuum X.*

Proof. Employ [8, Theorem 3.6, p. 100] to infer that there exist distinct points $p, q \in X$ such that $\{p\}, \{q\} \in NB^*(X)$. Now, if X is irreducible about $\{p, q\}$, then [8, Proposition 6.5, p. 104] implies that $\{p, q\} \in NB^*(X)$. Next, assume that there exists a proper subcontinuum Y of X containing both p and q. By [8, Theorem 3.6, p. 100], there exists $r \in X \setminus Y$ satisfying $\{r\} \in NB^*(X)$. In conclusion, $NB^*(X)$ has at least three elements. □

From [14, Theorem 2 and Corollary 1, p. 435], it follows that each one of the hyperspaces $S(X)$ and $NC(X)$ has at least two degenerate elements for any continuum X. The following result strengthens this fact and is an immediate consequence of Theorem 1.1 and Theorem 3.1 together.

Corollary 3.2. *Each one of the hyperspace $S(X)$, $NSC(X)$ and $NC(X)$ contains at least three elements for any continuum X.*

The hyperspaces $NB(X)$ and $NWC(X)$ cannot be included in the last result (see [3, Example 4.9, p. 37])

A routine argument establishes the following result, which is included here for sake of completeness. Part (1) follows from [11, (c) of Proposition 1.1, p. 654]; however, the proof presented here uses the terminology introduced in this paper.

Theorem 3.3. *Let X be a continuum. Each one of the following statements hold.*

(1) *Each nonempty closed subset of X contained in an element of $NB^*(X)$ belongs to $NB^*(X)$.*
(2) *Each nonempty closed subset of X contained in an element of $S(X)$ belongs to $S(X)$.*
(3) *Each nonempty closed subset of X contained in an element of $NSC(X)$ belongs to $NSC(X)$.*
(4) *Each nonempty closed subset of X contained in an element of $NC(X)$ belongs to $NC(X)$.*

Proof. Let $A \in NB^*(X)$ and let B be a nonempty closed subset of X contained in A. Observe that B is a nowhere dense subset of X and, if L is a continuum-wise connected component of $X \setminus A$ such that L is a dense subset of X, then the continuum-wise connected component of $X \setminus B$ containing L is a dense subset of X. This proves that $B \in NB^*(X)$. Hence, (1) holds.

In order to see (2), let $A \in S(X)$ and let B be a nonempty closed subset of X contained in A. There exists a subcontinuum L of $X \setminus A$ such that $H(L, X) < \varepsilon$. Then L is a subcontinuum of $X \setminus B$ satisfying $H(L, X) < \varepsilon$. Thus, $B \in S(X)$.

Now, let $A \in NSC(X)$, let B be a nonempty closed subset of X, and let U and V be nonempty open subsets of X. There exists a subcontinuum M of $X \setminus A$ such that $U \cap M \neq \emptyset$ and $V \cap M \neq \emptyset$. So, M is a subcontinuum of $X \setminus B$ satisfying $U \cap M \neq \emptyset$ and $V \cap M \neq \emptyset$. This proves that $B \in NSC(X)$. Therefore, (3) holds.

Finally, let $A \in NC(X)$ and let B be a nonempty closed subset of X. Let U and V be disjoint open subsets of X satisfying $X \setminus B = U \cup V$. Since the connected subset $X \setminus A$ of X is contained in $U \cup V$, either $X \setminus A \subseteq U$ or $X \setminus A \subseteq V$. This and the fact that $X \setminus A$ is a dense subset of X imply that either U is empty or V is empty. So, $X \setminus B$ is connected. □

The continuum presented in Example 3.8 shows that the hyperspaces $NWC(X)$ and $NB(X)$ are not closed under taking closed subsets.

Theorem 3.4. *If $X = u - irr(A)$, then every nonempty closed subset of X contained in A belongs to $NWC(X)$.*

Proof. Use Theorem 2.4 to infer that $\eta(A) = X \setminus A$. Now, let B a nonempty closed subset of X contained in A. Fix $x \in \eta(A)$. Let $y \in X \setminus B$. Assume that $y \in \eta(A)$. From Theorem 2.5, it follows that there exists a subcontinuum of $\eta(A)$ containing both x and y. Then a subcontinuum of $X \setminus B$ contains both x and y. Now, suppose that $y \in A$. Proposition 2.6 guarantees the existence of a subcontinuum Y of X such that $x \in Y$ and $Y \cap A = \{y\}$. Then Y omits B. So, Y is a subcontinuum of $X \setminus B$ containing both x and y. In conclusion, $X \setminus B$ is continuum-wise connected. □

Apply Theorem 1.1 and Theorem 3.4 together to deduce the following result.

Corollary 3.5. *If $X = u - irr(A)$, then every nonempty closed subset of X contained in A belongs to $NB(X)$, $NB^*(X)$, $S(X)$, $NSC(X)$, and $NC(X)$.*

The last result implies [14, Theorem 3 and Proposition 2, pp. 346 and 348] and [8, Proposition 6.3 and Corollary 6.4, p. 104]

Theorem 3.6. *If $X = c - irr(A)$, then every nonempty closed subset of X contained in $X \setminus \eta(A)$ belongs to $NB^*(X)$.*

Proof. Let B be a nonempty closed subset of X contained in $X \setminus \eta(A)$. Now, Theorem 2.5 proves that $\eta(A)$ is a continuum-wise connected dense subset of X. This implies that the continuum-wise connected component of $X \setminus B$ containing $\eta(A)$ is a dense subset of X. So, $B \in NB^*(X)$. □

The following result is an immediate consequence of Theorem 1.1 and Theorem 3.6 together.

Corollary 3.7. *If $X = c - irr(A)$, then every nonempty closed subset of X contained in $X \setminus \eta(A)$ belongs to $S(X)$, $NSC(X)$, $NC(X)$.*

The continuum presented in Example 3.8 shows that the hyperspaces $NWC(X)$ and $NB(X)$ cannot be included in Corollary 3.7.

Example 3.8. Let X be a compatification of a ray with nondegenerate remainder I. Note that $I \in NWC(X)$. Let $p, q \in X$ be such that $X = irr(\{p, q\})$. Assume that $p \in I$. Hence, $\lambda(p, \{p, q\}) = I$ and $\lambda(q, \{p, q\}) = \{q\}$. Then $X = c - irr(\{p, q\})$. On the other hand, if B is a proper nonempty closed subset of I, then a continuum-wise connected component of $X \setminus B$ is contained in I and this is not a dense subset of X. This argues that $B \notin NB(X)$.

Recall that for a family of sets J, the symbol $\bigcup J$ is the set of all points contained in an element of J.

Theorem 3.9. *If $X = c - irr(A)$ and $\bigcup NB^*(X)$ is at most countable, then $X = u - irr(A)$.*

Proof. Invoke Theorem 3.6 to deduce that $\lambda(a, A)$ is member of $NB^*(X)$ for each $a \in A$. Apply Proposition 3.3 to conclude that $X \setminus \eta(A)$ is a subset of $\bigcup NB^*(X)$. Hence, $X \setminus \eta(A)$ is at most countable. This implies that each subcontinuum $\lambda(a, A)$ of X must be degenerate. From this and the fact that $a \in \lambda(a, A)$, it follows that $\lambda(a, A) = \{a\}$. Apply Theorem 2.4 to conclude that $X = u - irr(A)$. □

Corollary 3.10. *If $X = c - irr(A)$ and one of $\bigcup S(X)$, $\bigcup NSC(X)$, and $\bigcup NC(X)$ is at most countable, then $X = u - irr(A)$.*

Let X be the continuum presented in [3, Example 4.10, p. 38]. Then $NB(X)$ consists uniquely of singletons and is homeomorphic to the closure of a convergent sequence. Set $A = \{x \in X : \{x\}$ is an isolated point of $NB(X)\}$. Then $X = u - irr(A)$ and $\bigcup NB(X)$ is countable. This arises the following question.

Question 3.11. *Under the assumption $X = c - irr(A)$, does the condition $\bigcup NB(X)$ is countable imply $X = u - irr(A)$?*

The following result is an immediate consequence from Theorems 1.1, 3.4, and 3.9.

Corollary 3.12. *Let X be an irreducible continuum about a nonempty subset A of X such that $\lambda(a, A)$ is a subcontinuum of X for each $a \in A$. The implications $(i) \Rightarrow (j)$ when $i < j$ hold for the following conditions.*

(1) *$\bigcup NC(X)$ is at most countable.*
(2) *$\bigcup NSC(X)$ is at most countable.*
(3) *$\bigcup S(X)$ is at most countable.*
(4) *$\bigcup NB^*(X)$ is at most countable.*
(5) *$X = u - irr(A)$.*
(6) *$\{B \in 2^X : B \subseteq A\} \subseteq NWC(X)$.*
(7) *$\{B \in 2^X : B \subseteq A\} \subseteq NB(X)$.*
(8) *$\{B \in 2^X : B \subseteq A\} \subseteq NB^*(X)$.*
(9) *$\{B \in 2^X : B \subseteq A\} \subseteq S(X)$.*
(10) *$\{B \in 2^X : B \subseteq A\} \subseteq NSC(X)$.*
(11) *$\{B \in 2^X : B \subseteq A\} \subseteq NC(X)$.*

Lemma 3.13. *If $X = c - irr(A)$ and A is finite, then each member of $NB^*(X)$ intersects $X \setminus \eta(A)$.*

Proof. Suppose to the contrary that there exists $B \in NB^*(X)$ contained in $\eta(A)$. Let L be the continuum-wise connected component of $X \setminus B$ such that L is a dense subset of X.

Now, let $a \in A$. Then $\lambda(a, A)$ is a subcontinuum of X contained in the open subset $X \setminus B$ of X. Choose a subcontinuum $E(a)$ of X containing properly $\lambda(a, A)$ and contained in $X \setminus B$ guaranteed by [17, Corollary 5.5, p. 74]. From Proposition 2.2, it follows that $\lambda(a, A)$ is a subset of the interior of $E(a)$. Then L intersects the interior of $E(a)$. Since L is continuum-wise connected component of $X \setminus B$, $E(a)$ is a subset of L. Hence, L contains a subcontinuum M of X such that the finite collection of subcontinua $\{E(a) : a \in A\}$ of X is a subset of M. Then A is a subset of M and M is a subset of $X \setminus B$. This contradicts the fact that $X = irr(A)$. Therefore, $B \setminus \eta(A) \neq \emptyset$. □

Compare the next result with [6, (1) of Lemma 4.4, p. 75].

Theorem 3.14. *If $X = c - irr(A)$ and A is finite, then each member of $NB^*(X)$ is contained in $X \setminus \eta(A)$.*

Proof. Let $B \in NB^*(X)$ and let $y \in B$. Apply Proposition 3.3 to infer that $\{y\} \in NB^*(X)$. By Lemma 3.13, y is an element of $X \setminus \eta(A)$. Then B is a subset of $X \setminus \eta(A)$. □

Corollary 3.15. *If $X = c - irr(A)$ and A is finite, then each member of $NWC(X)$ and of $NB(X)$ is contained in $X \setminus \eta(A)$.*

The following example shows that the condition requiring that A to be finite in Theorem 3.14 is necessary.

Example 3.16. For each $n \in \mathbb{N}$, let L_n denote the line segment in the Euclidean plane joining $(0, 0)$ and $\left(1, \frac{1}{n}\right)$. Let $L_0 = \{0\} \times [0, 1]$. Set $X = \bigcup_{n=0}^{\infty} L_n$ and $A = \left\{\left(1, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$. Then X is a continuum such that $X = irr\,(A)$ and $\lambda(a, A) = \{a\}$ for each $a \in A$. Notice that each nonempty closed subset of X contained in $X \setminus \bigcup_{n \in \mathbb{N}} L_n$ is an element of $NB^*(X)$ and is disjoint from $\bigcup_{a \in A} \lambda(a, A)$.

Combine Theorem 3.6 with Theorem 3.14 to infer the following result.

Corollary 3.17. *If $X = c - irr(A)$ and A is finite, then a nonempty closed subset B of X is a member of $NB^*(X)$ if and only if B is a subset of $X \setminus \eta(A)$.*

The following result answers [8, Question 6.13, p. 106] in the positive form and, hence, improves [8, Theorem 6.12, p. 106] and [14, Proposition 2, p. 438].

Theorem 3.18. *If $X = u - irr(A)$ and A is finite, then $NWC(X) = NB(X) = NB^*(X) = S(X) = \{B \in 2^X : B \subseteq A\}$.*

Proof. Apply [8, Theorem 6.11, p. 105] to obtain that $S(X) \subseteq \{B \in 2^X : B \subseteq A\}$. Since $NB^*(X)$ is a subset of $S(X)$ and A is finite, $NB^*(X)$ is finite. This and [6, Theorem 3.2, p. 73] together guarantee that $NWC(X) = NB(X) = NB^*(X) = S(X)$. Theorem 3.4 guarantees the inclusion $\{B \in 2^X : B \subseteq A\} \subseteq NWC(X)$. This ends the proof. □

The following results follows immediately from Corollaries 3.12 and, Theorems 3.14 and 3.18.

Corollary 3.19. *If $X = c - irr(A)$ and A is finite, then the following conditions are equivalent.*

(1) $3 \leq |NB^*(X)| < \infty$.
(2) $X = u - irr(A)$.
(3) $NWC(X) = NB(X) = NB^*(X) = S(X) = \{B \in 2^X : B \subseteq A\}$.

The final results of this paper require the property of Kelley.

Theorem 3.20. *Let X be a continuum having the property of Kelley. If $B \in NB^*(X)$ and X is c.i.k. at some point of $X \setminus B$, then $B \in NWC(X)$.*

Proof. Assume that X is c.i.k. at $p \in X \setminus B$. Let K be the continuum-wise connected component of $X \setminus B$ containing p. Lemma 2.7 guarantees that K is an open subset of X. Now, let L be the continuum-wise connected component of $X \setminus B$ such that L is a dense subset of X. Hence, $L \cap K \neq \emptyset$. This implies that $K = L$. So, $X \setminus B$ is continuum-wise connected. In other words, $B \in NWC(X)$. □

Corollary 3.21. *Let X be a continuum having the property of Kelley. If every dense open subset of X intsercts $\{x \in X : X$ is c.i.k. at $x\}$, then $NB(X) = NWC(X)$.*

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