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Dissemination and Research in the Study of Complex Systems and their Applications

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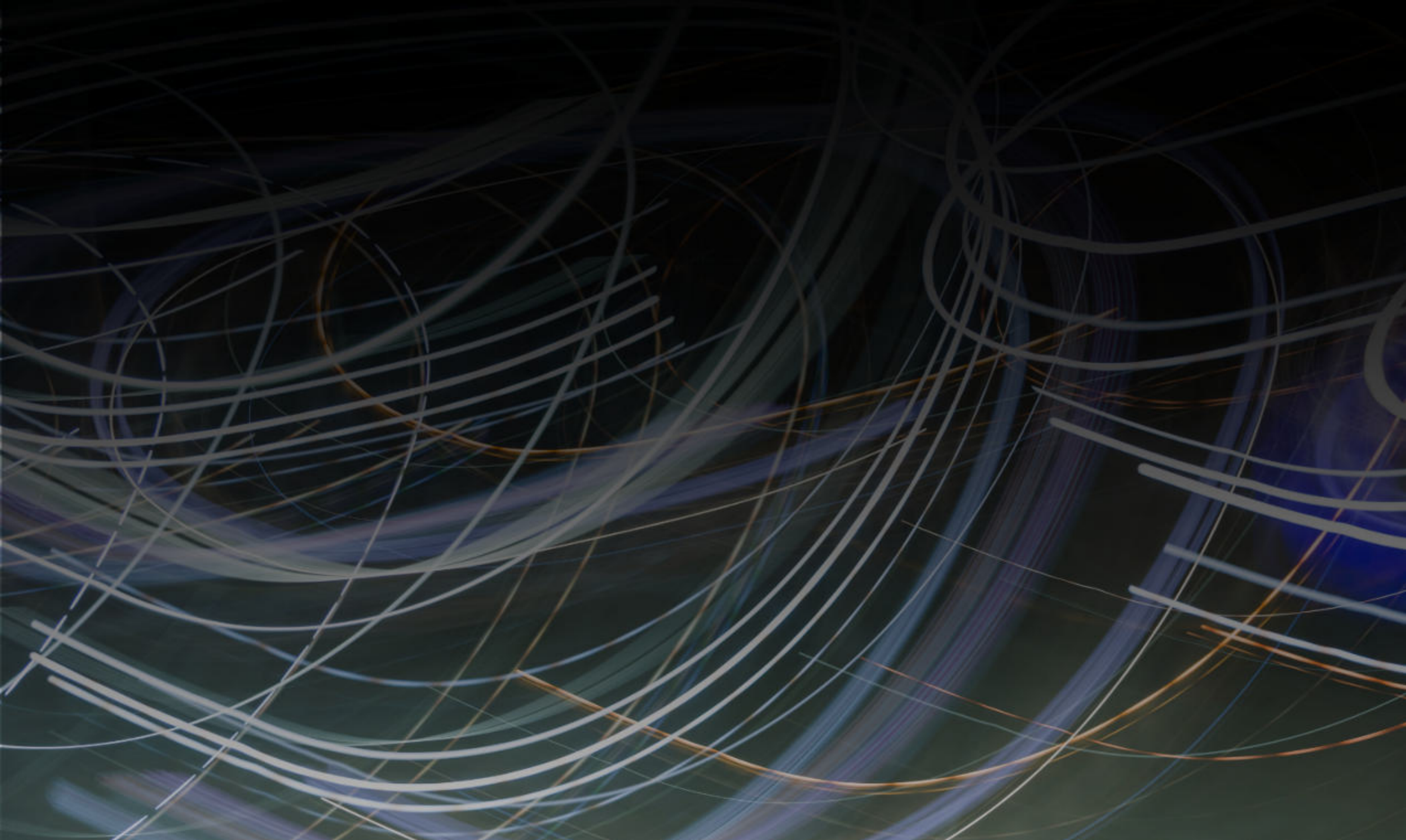

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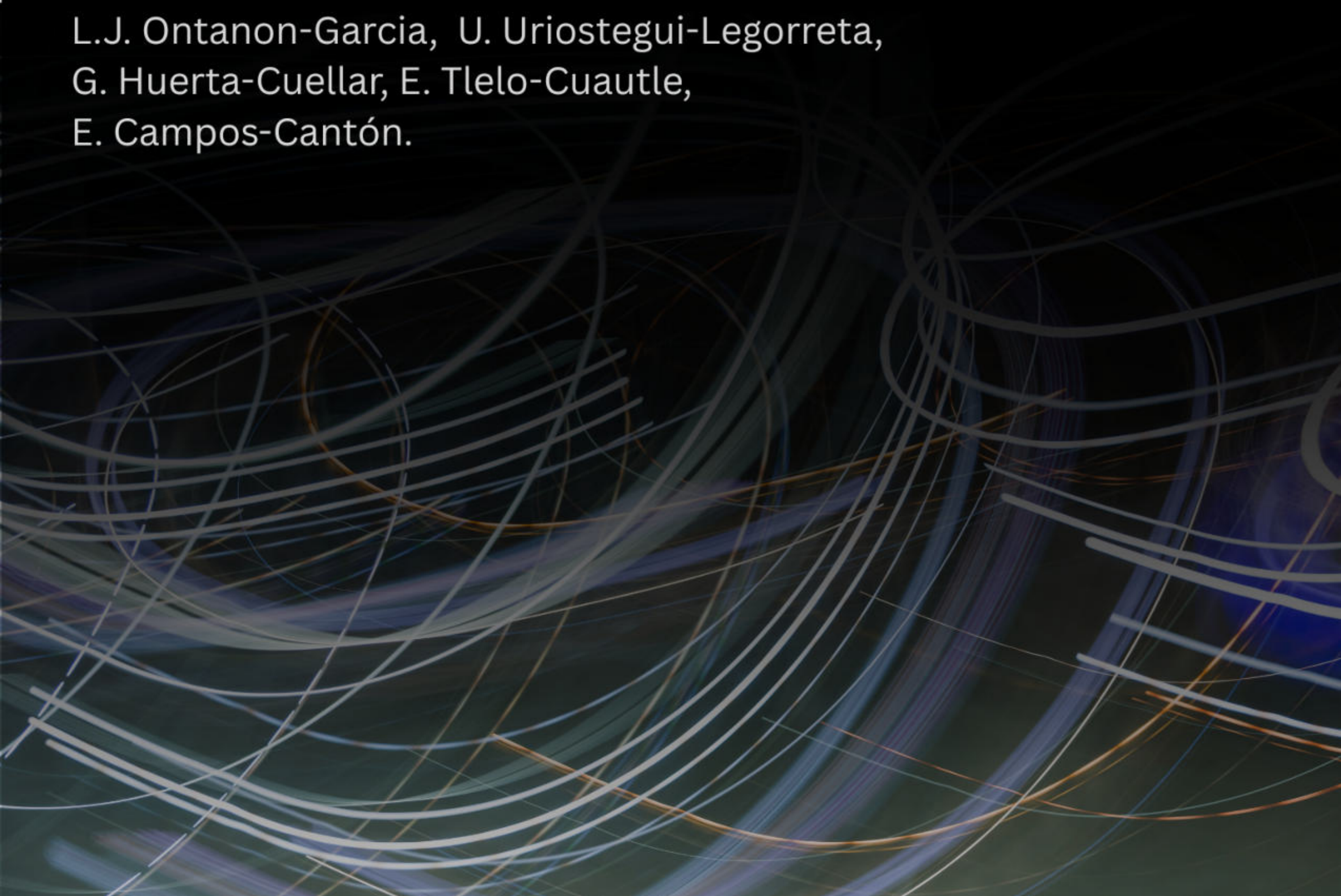

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Chapter 11. Wavelet Resolution Analysis of State Variables in a Double-Scroll Chaotic Attractor

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Abstract

This work investigates the distribution of signal energy across resolution levels for the three state variables (x_1, x_2, x_3) *of the double-scroll attractor using discrete wavelet transforms. The analysis identifies, for each state, the resolution levels that concentrate the largest fraction of the signal energy, thus revealing which variables and scales are most informative for reconstructing the chaotic trajectory. The results show that the states differ in their energy localization, suggesting variable-specific roles in encoding the system's dynamics. Although this study is limited to the double-scroll configuration, we hypothesize that in multi-scroll chaotic systems, the state directly responsible for scroll generation may exhibit a broader, possibly fractal-like, energy distribution, while other states remain more concentrated in a small set of resolution levels. This insight opens the possibility of efficient state selection and scale targeting for chaotic signal recovery, masking, or feature extraction in future work.*

Keywords: Wavelet transform, chaotic time series, double-scroll attractor, multi-resolution analysis, energy distribution

1 Introduction

Context and scope. This contribution examines the double–scroll attractor as a foundational case study within a broader program on wavelet-based analysis of multi-scroll chaotic systems. Focusing on this canonical example allows us to articulate the methodology, quantify how wavelet energy localizes across resolution levels for each state variable, and establish a baseline for subsequent extensions to higher-scroll configurations.

Chaotic oscillators often exhibit rich and complex dynamics that can be decomposed into multiple state variables, each contributing differently to the evolution of the system. Among them, the double-scroll attractor is a paradigmatic example in nonlinear science, widely studied for its distinctive two-lobed structure and potential applications in secure communications, signal processing, and feature extraction. Despite its apparent symmetry, the states (x_1, x_2, x_3) of the double-scroll attractor do not contribute equally to the overall trajectory; some states may carry more reconstructive or predictive information than others.

Wavelet transforms offer a natural framework for analyzing chaotic signals due to their ability to localize both time and frequency content across multiple resolution levels. By examining the energy distribution of a state's wavelet coefficients, one can identify which resolution levels dominate its representation, thus revealing scale-specific features that are relevant for signal recovery or analysis.

The present work focuses on a detailed examination of the three states of the double-scroll attractor, quantifying the concentration of wavelet energy at different resolution levels. The results allow us to determine, for each state, the scales that capture most of its signal energy and therefore its contribution to trajectory reconstruction.

While the analysis is restricted to the double-scroll configuration, the conceptual goal extends further: in systems with more than two scrolls, we hypothesize that the state responsible for generating additional scrolls may exhibit a more distributed, potentially fractal-like, wavelet energy profile, whereas the remaining states would retain strong localization at a smaller set of resolution levels. Confirming this behavior could lead to systematic methods for identifying optimal states and scales for chaotic signal recovery, masking, or feature extraction in higher-dimensional and more complex chaotic systems.

2 Wavelet Transform

The wavelet transform (WT) provides a time–frequency representation of signals, complementing the Fourier transform [1]. It is particularly useful for analyzing non-stationary and chaotic signals [2].

The **continuous wavelet transform** (CWT) of a function $f(t)$ is obtained using the scaled and translated mother wavelet $\psi(t)$:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right), \tag{1}$$

where $a > 0$ is the scale (dilation) parameter and $b \in \mathbb{R}$ is the translation.

For computational purposes, parameters are discretized dyadically:

$$a = 2^{-j}, \quad b = k2^{-j}, \tag{2}$$

leading to the **discrete wavelet transform** (DWT) basis:

$$\psi_{j,k}(t) = 2^{j/2}\,\psi(2^j t - k). \tag{3}$$

The DWT coefficients are computed as:

$$f_{j,k} = \int_{-\infty}^{\infty} f(t)\,\psi_{j,k}(t)\,dt, \tag{4}$$

allowing the signal reconstruction via the orthonormal expansion:

$$f(t) = \sum_j \sum_k f_{j,k}\,\psi_{j,k}(t). \tag{5}$$

In **multi-resolution analysis** (MRA), the scaling function $\varphi(t)$ and wavelet $\psi(t)$ satisfy:

$$\varphi(t) = \sqrt{2}\sum_k h_k\,\varphi(2t-k), \tag{6}$$

$$\psi(t) = \sqrt{2}\sum_k g_k\,\varphi(2t-k), \tag{7}$$

where $g_k = (-1)^k h_{1-k}$. This decomposition hierarchically separates a signal into approximation and detail subspaces.

Many chaotic and physical signals exhibit **self-similarity**, expressed as:

$$f(t) \cong a^{-H} f(at), \tag{8}$$

with H the self-similarity parameter. For such signals, the variance of DWT coefficients scales with:

$$\mathrm{var}(f_{j,k}) \propto (2^j)^{-\beta}, \quad \beta = 2H+1. \tag{9}$$

This property aids in distinguishing noise-like from structured dynamics, as later illustrated.

3 Chaotic Attractor with Multiple Scrolls

We consider a class of linear systems capable of generating chaotic attractors [3, 4], described by:

$$\dot{x} = Ax + B, \tag{10}$$

where $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ is the state vector, $B = [0, 0, \beta_3]^T$, and $A \in \mathbb{R}^{3\times 3}$ is:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix}. \tag{11}$$

The parameters $\alpha_{31}, \alpha_{32}, \alpha_{33}$ are chosen to ensure piecewise linearity and chaotic behavior. In this work, we use $\alpha_{31} = 1.5$, $\alpha_{32} = 1$, and $\alpha_{33} = 1$.

A double-scroll attractor is obtained by introducing a switching control signal β_3 that alternates between two values depending on x_1:

$$\beta_3 = \begin{cases} 1.8, & \text{if } x_1 \geq 0.3, \\ -0.9, & \text{otherwise.} \end{cases} \tag{12}$$

This yields two distinct equilibrium points, $x_0^{*1} = (0.6, 0, 0)^T$ and $x_0^{*2} = (0, 0, 0)^T$, producing the characteristic double-scroll chaotic attractor depicted in Figure 1(d).

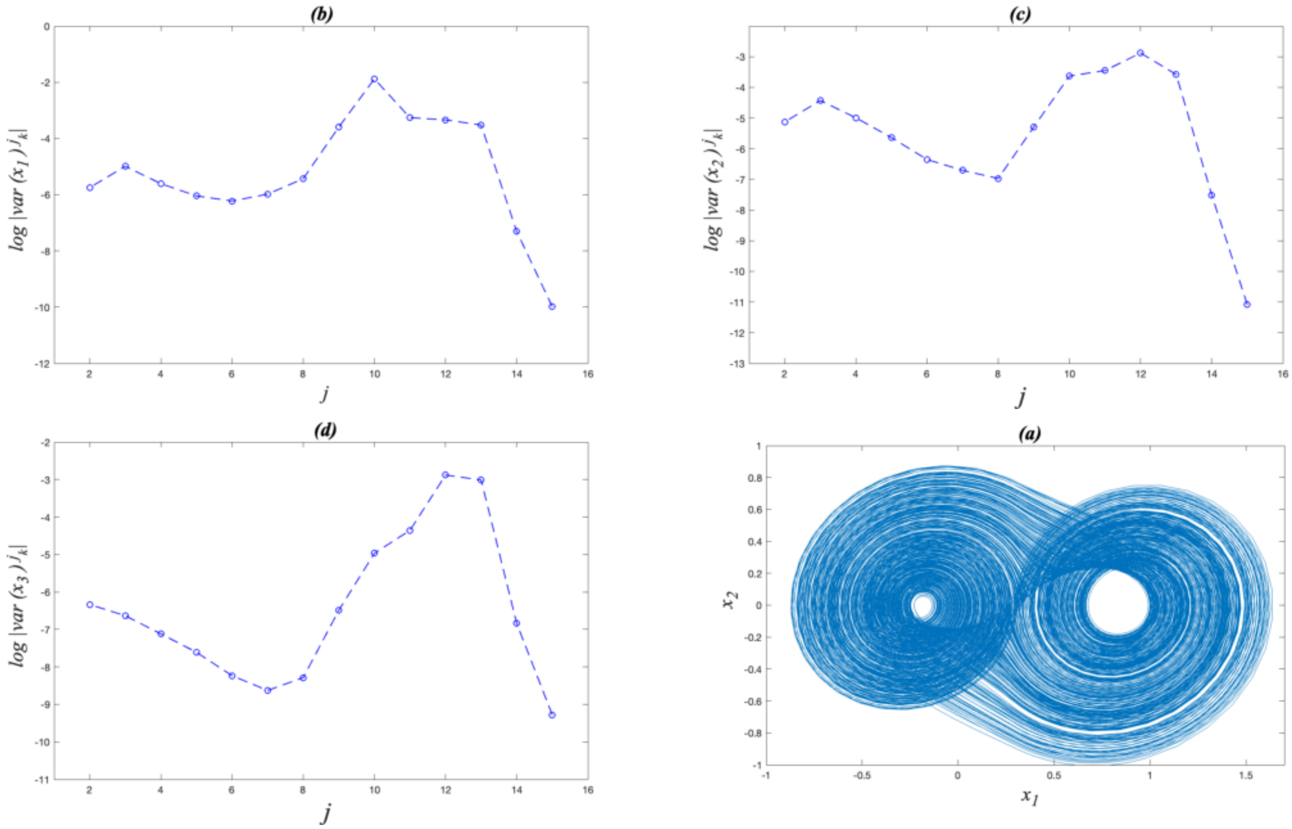

Figure 1: Wavelet-based analysis for the double-scroll system. (a) x_1; (b) x_2; (c) x_3; (d) projection of the double-scroll attractor onto (x_1, x_2).

4 Analysis of Chaotic Time Series

We apply the discrete wavelet transform (WT) to the chaotic time series generated by the double-scroll system described in Section 3. The analysis focuses on the energy distribution of wavelet coefficients across resolution levels, which reveals dominant frequency components, noise-like regions, or fractal behavior depending on the slope of the semi-logarithmic variance curve from Eq. (9). The following observations clarify how such behavior can be characterized:

1. A peak in the semi-logarithmic variance graph indicates a high concentration of energy at a specific level, suggesting a carrier frequency in the time series.
2. A horizontal line (zero slope) suggests an even energy distribution, indicating Gaussian noise behavior.
3. A negative slope implies fractal behavior in the time series.

These behaviors can be interpreted through the logarithm of the variance as a function of resolution levels, based on equation (9), which involves the coefficients $f_{j,k}$, the resolution levels j, and the self-similarity exponent β.

Figure 1(a) presents the wavelet analysis of the state variable x_1 for the double-scroll system, where the energy is concentrated at level $j = 10$, indicating the presence of a dominant frequency component. Figure 1(b) shows the analysis for state x_2, where the system exhibits significant energy at level $j = 12$. Figure 1(c) displays similar results for state x_3, with dominant energy at levels $j = 12$ and $j = 13$, which jointly enable an accurate signal representation. In all cases, the double-scroll system exhibits precise frequency localization rather than the more distributed patterns expected in higher-scroll configurations.

5 Discussion and Future Work

The present analysis focused on the three state variables (x_1, x_2, x_3) of the double-scroll attractor, identifying the resolution levels that capture most of the signal energy for each state. These findings raise relevant questions for systems with more than two scrolls:

- How does increasing the number of scrolls affect the wavelet energy distribution of each state?
- Which state retains the highest concentration of reconstructive information as system complexity grows?
- At which resolution levels is this information most effectively localized for accurate trajectory reconstruction?

Our working hypothesis is that, as scroll count increases, the state directly responsible for scroll generation (x_1 in our case) may exhibit a more distributed or fractal-like energy profile, while the remaining states maintain stronger localization around a few resolution levels. Identifying these optimal states and levels would enable efficient signal recovery, masking, or feature extraction in chaotic systems. Future work will extend the numerical study to higher-scroll configurations, assess robustness under noise, and evaluate potential applications in secure communications and pattern recognition using wavelet-based signatures.

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